H.S.C. CORNER BY TREVOR

This term we will concentrate on complex numbers, and solve problems from the 1981 and 1982 4-unit papers.

De Moivre's theorem states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n \text{ any integer.}$$

This result may be proved using induction, and extended to

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

where p,q are integers.

We now consider two problems. Firstly from 1982:

<u>Problem 83.5</u>. Express $\tan 5\theta$ as a rational function of $t = \tan\theta$. Hence deduce that

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5.$$

<u>Solution</u>: For $n = 5$, use the binomial theorem to show that

$$\begin{aligned}\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta \\ &\quad + 5\cos\theta\sin^4\theta + i\sin^5\theta.\end{aligned}$$

In any complex equation, both the real and imaginary parts must be separately equal. Thus:

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \tag{1}$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta. \tag{2}$$

Divide (2) by (1), and divide numerator and denominator by $\cos^5\theta$, to obtain

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4},$$

where $t = \tan\theta$. Now consider the equation $\tan 5\theta = 0$. For $0 \leqslant \theta < \pi$ this has the five solutions $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$, i.e. $\theta = \frac{n\pi}{5}$, $n = 0, 1, \ldots, 4$. Thus $\tan\frac{n\pi}{5}$ must be a solution of the equation

$$t(5 - 10t^2 + t^4) = 0.$$

The solution $t = 0$ corresponds to $n = 0$, and $t = \tan\frac{n\pi}{5}$, $n \neq 0$, are the solution of the quartic equation in the bracket. The product of the four roots is 5, and hence

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5.$$

Now try this: Problem 83.5a (solution next issue).

Show also that:

(i) $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{3\pi}{5}\cos\frac{4\pi}{5} = \frac{5}{16}$

(ii) $\cos^2\frac{\pi}{5} + \cos^2\frac{2\pi}{5} = \frac{5}{4}$.

A related problem was set in 1981:

Problem 83.6. Let $\omega (\neq 1)$ be a solution of $x^3 = 1$.

(a) Show that ω^2 is also a root.

(b) Show that $1 + \omega + \omega^2 = 0$, and $1 + \omega^2 + \omega^4 = 0$.

(c) Let α, β be real numbers. Find, in its simplest form, the equation whose roots are

$$\alpha + \beta,\quad \alpha\omega + \beta\omega^{-1},\quad \alpha\omega^2 + \beta\omega^{-2}.$$

Solution:

(a) $$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

$\therefore$ the roots of $x^3 - 1 = 0$ are $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Hence, let $\omega = \frac{1}{2}(-1 \pm \sqrt{3}i)$, then $\omega^2 = \frac{1}{4}(1 - 3 \mp \sqrt{3}i) = \frac{1}{2}(-1 \mp \sqrt{3}i)$. Thus ω^2 is also a root of the equation.

(b) Thus the sum of the roots is of the equation $x^3 - 1 = 0$ is $1 + \omega + \omega^2$, and hence $1 + \omega + \omega^2 = 0$. Also $1 + \omega^2 + \omega^4 = 1 + \omega^2 + \omega(\omega^3) = 1 + \omega^2 + \omega = 0$.

(c) Sum of roots is $\alpha(1 + \omega + \omega^2) + \beta(1 + \frac{1}{\omega} + \frac{1}{\omega^2}) = (\alpha + \frac{\beta}{\omega^2})(1 + \omega + \omega^2) = 0$.

Also the product of roots is

$$(\alpha + \beta)(\alpha\omega + \beta\omega^{-1})(\alpha\omega^2 + \beta\omega^{-2}) = \alpha^3\omega^3 + \alpha^2\beta(\omega^{-1} + \omega + \omega^3) + \alpha\beta^2(\omega^{-3} + \omega^{-1} + \omega) + \beta^3\omega^{-3}$$

$$= \alpha^3 + \beta^3$$

and the sum of products in pairs is

$$(\alpha + \beta)(\alpha\omega + \beta\omega^{-1}) + (\alpha\omega + \beta\omega^{-1})(\alpha\omega^2 + \beta\omega^{-2}) + (\alpha\omega^2 + \beta\omega^{-2})(\alpha + \beta)$$

$$= \alpha^2(\omega + \omega^3 + \omega^2) + \alpha\beta(\omega + \omega^{-1} + \omega + \omega^{-1} + \omega^2 + \omega^{-2}) + \beta^2(\omega^{-1} + \omega^{-3} + \omega^{-2})$$

$$= \alpha^2\omega(1 + \omega + \omega^2) + \omega^{-2}\alpha\beta(3 + 3\omega) + \beta^2\omega^{-3}(1 + \omega + \omega^2)$$

$$= -3\alpha\beta.$$

Thus the required cubic equation is

$$y^3 + 0y^2 - 3\alpha\beta y - (\alpha^3 + \beta^3) = 0$$

i.e. $\underline{y^3 - 3\alpha\beta y - \alpha^3 - \beta^3 = 0.}$ (A)

Extension. The result in (A) is the essential step of Cardan's solution of the cubic equation $y^3 + ax + b = 0$. For example, consider the cubic $y^3 + 3y + 2 = 0$. Comparing with (A),

$$\alpha^3 + \beta^3 = -2, \quad \alpha\beta = -1,$$

whence

$$\alpha^3 - \frac{1}{\alpha^3} = -2$$

and

$$\alpha^6 + 2\alpha^3 - 1 = 0, \quad \text{so that} \quad \alpha^3 = -1 + \sqrt{2} \qquad \beta^3 = -1 - \sqrt{2}$$

and $\underline{\alpha = (\sqrt{2} - 1)^{1/3}}$, $\underline{\beta = -(\sqrt{2} + 1)^{1/3}}$. By the assumption used to obtain (A), the required roots are:

$$\alpha + \beta, \quad \alpha\omega + \beta\omega^{-1}, \quad \alpha\omega^2 + \beta\omega^{-2} \;!$$