THE 1983 SCHOOL MATHEMATICS COMPETITION

JUNIOR DIVISION

Q1. 6 positive whole numbers A,B,C,D,E and F satisfy $A + B = 29$, $C + D = 45$, $E + F = 65$, $AC = 36$ and $BE = 312$. Find A,B,C,D,E and F. Show there is only one solution to the equations.

Solution. The equation $BE = 312$ shows that B and E must be divisors of $312 = 2^3.3.13$ and so they must come from the set {1, 2, 4, 8, 3, 6, 12, 24, 13, 26, 52, 104, 39, 78, 156, 312}. The equations $A + B = 29$ and $E + F = 65$ show that B must be less than 29 and E must be less than 65. After this is taken into account, the possible values for B and E are

$$B = 8 \quad 6 \quad 12 \quad 24 \quad 13 \quad 26$$
$$E = 39 \quad 52 \quad 26 \quad 13 \quad 24 \quad 12.$$

From the equation $A + B = 29$, the corresponding values of A are

$$A = 21 \quad 23 \quad 17 \quad 5 \quad 16 \quad 3.$$

But the equation $AC = 36$ shows that A must divide 36 exactly, so the only possible value for A is $A = 3$. This gives the unique solution

$$A = 3, \quad B = 26, \quad C = 12, \quad D = 33, \quad E = 12, \quad F = 53.$$

Q2. Write the numbers 1, 0 or -1 in the squares of a 3×3 table and add up the numbers in each row and each column. Is it true that among the 6 numbers so obtained there must be two which are equal? Prove your assertion.

Solution. Suppose the six row and column sums are all different. The possible values for these sums are -3, -2, -1, 0, 1, 2, 3, and our assumption means that six of these seven numbers appear as row and column sums. The missing number must be even, namely 0, 2 or -2. To see this, note that the sum of the three row sums is equal to the sum of the three column sums, so the total of all the six row and column sums is even. Now some row or column adds to 3; say the first row contains three

1's. Also some row or column adds to -3; say the second row contains three -1's. The entries in the third row must now be distinct to ensure that the column sums are all different; say the last row is $-1, 0, 1$. But, after all this, the third row and the second column both sum to 0. So the six row and column sums cannot all be different.

1	1	1	3
-1	-1	-1	-3
-1	0	1	0
-1	0	1	

Q3. When x,y are any whole numbers, $x \# y$ denotes another whole number. The operation $\#$ has the following properties:

(i) $x \# 0 = x$ for all x

(ii) $0 \# y = -y$ for all y

(iii) $((x+1) \# y) + (x \# (y+1)) = 3(x \# y) - xy + 2y$

for all x,y.

Find the value of $19 \# 83$.

Solution. First a little experimentation. From (i), $x \# 0 = x$. From (iii), with $y = 0$,

$$x \# 1 = 3(x \# 0) - (x+1) \# 0 = 3x - (x+1) = 2x - 1.$$

Again, from (iii) with $y = 1$,

$$x \# 2 = 3(x \# 1) - (x+1) \# 1 - x + 2 = 3(2x-1) - (2(x+1) - 1) - x + 2 = 3x - 2.$$

After a bit more of this, we guess that

$$x \# y = x(y+1) - y = xy + x - y.$$

It is then easy to verify that this formula for $x \# y$ does have the properties (i), (ii) and (iii). These properties enable us to calculate $x \# 1$, $x \# 2$, $x \# 3$, ... in turn, as already indicated, so $x \# y$ is uniquely determined by (i), (ii) and (iii). Thus our guess is the only possible formula for $x \# y$ and

$$19 \# 83 = 19 \times 83 + 19 - 83 = 1513.$$

Q4. In a certain boy's school everyone who played either tennis or cricket either played football or didn't play chess. Everyone who either played chess or did not play cricket, either played tennis or did not play football. No-one who played either football or cricket played both chess and tennis. Jim Adams played chess. Which of the other games, if any, might he have played?

Solution. Jim Adams played chess. Suppose he also played tennis. From the first sentence of the problem, he must have played football. From the third sentence, he did not play tennis. So Jim Adams did not play tennis. From the second sentence, he did not play football. Finally from the first sentence, he did not play cricket. He sounds like a bit of a swot.

Q5. A new piece has been invented for use in the game of chess. This piece, the duke, makes moves rather similar to those made by the knight. We call the square in the rth row and the sth column of the chess board (r,s); a duke situated in this square may move to any of the eight squares $(r \pm 3, \ s \pm 1)$ or $(r \pm 1, \ s \pm 3)$ which are within the chess board. If the duke is in the nearest left-hand corner, i.e. in square $(1,1)$ and there are no other pieces on the board is it possible to move it to any of the other corner squares $(1,8)$, $(8,8)$ or $(8,1)$? If so, state in each case the minimum number of moves.

Solution. The duke starts in the bottom left-hand corner, a black square. Thereafter, he must always move on black squares, so he cannot reach the white corner squares $(1,8)$ and $(8,1)$. The sequence of moves

$$(1,1) \to (4,2) \to (7,3) \to (8,6) \to (5,7) \to (8,8)$$

gets the duke to the square $(8,8)$ in five moves. Note that he starts on a square with odd coordinates, then after one move is on a square with even coordinates, then after two moves is on a square with odd coordinates, and so on. So he must take an odd number of moves to reach $(8,8)$. It is easy to see that he cannot travel from $(1,1)$ to $(8,8)$ in three moves. To change the first coordinate by 7 in three moves, we need to add 3,3 and 1 in turn. The corresponding changes in the second coordinate are ±1, ±1 and ±3, but these can never add to 7. So the minimum number of moves to get from $(1,1)$ to $(8,8)$ is 5.

Q6. $0 < a_1 < a_2 < \ldots < a_n$ are natural numbers. Prove that their least common multiple is at least na_1.

Solution. We are asked for the smallest least common multiple attached to the n given numbers $a_1, \ldots, a_n$. Turning this around, we can start with the least common multiple, say N, and look for the n biggest divisors of N which will correspond to the numbers $a_n, a_{n-1}, \ldots, a_1$. Now the largest values we can assign to these divisors are

$$a_n = N/1, \quad a_{n-1} = N/2, \quad a_{n-2} = N/3, \ldots, a_1 = N/n.$$

In general, if $1, 2, 3, \ldots, n$ do not all divide N, then $a_1 \leqslant N/n$, that is $N \geqslant na_1$. So the least common multiple of the numbers $a_1, \ldots, a_n$ is at least na_1.

SENIOR DIVISION

Q1. The surface of a cylinder consists of one curved and 2 flat sections whereas that of a cone consists of one curved and one flat section. Suppose a right cone and a cylinder have a common circular face and the vertex of the cone is the centre of the opposite face of the cylinder. Suppose the ratio of their surface areas is $7 : 4$. Find the ratio of the length of the cylinder to its base radius.

Solution. Suppose the circular base of the cone and cylinder has radius r and the height of each is h. The surface area of the cylinder is $2\pi rh + 2\pi r^2$ and the surface area of the cone is $\pi r\sqrt{r^2 + h^2} + \pi r^2$. The ratio of these two quantities is $7 : 4$, so

$$\frac{2\pi rh + 2\pi r^2}{\pi r\sqrt{r^2 + h^2} + \pi r^2} = \frac{7}{4}.$$

In terms of $x = \frac{h}{r}$, this becomes

$$4(2x + 2) = 7(\sqrt{1 + x^2} + 1),$$

that is

$$8x + 1 = 7\sqrt{1 + x^2}$$

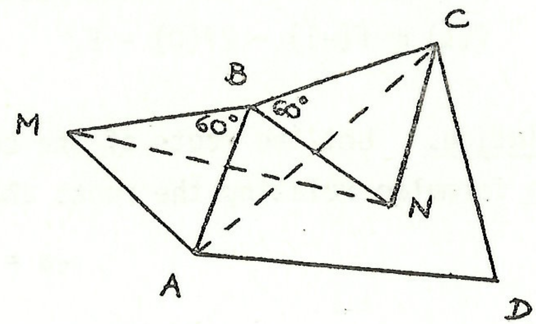
(continued over)

Squaring both sides gives the quadratic

$$15x^2 + 16x - 48 = (3x - 4)(5x + 12) = 0.$$

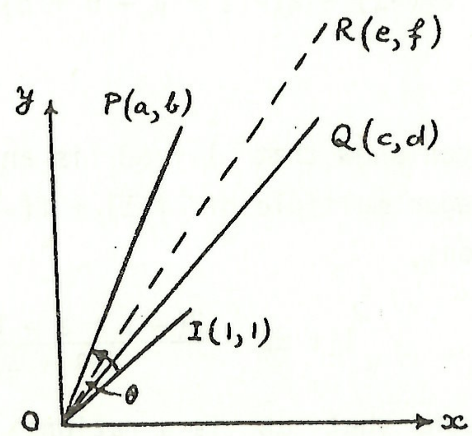
So $x = h/r = 4/3$.

Q2. Let ABCD be a convex quadrilateral, and draw equilateral triangles ABM, CDP, BCN, ADQ to the sides, the first two outwards, the other two inwards. Prove that MN = AC. What can you say about the quadrilateral MNPQ?



Solution. The triangles ABM and BCN are shown in the figure. In triangles ABC and MBN, AB = MB (being sides of equilateral triangle ABM), BC = BN (being sides of equilateral triangle BCN) and $\angle ABC = \angle MBN$ (both being 60° plus $\angle ABN$). So triangles ABC and MBN are congruent and AC = MN. Similarly AC = PQ and BD = NP = MQ. The quadrilateral MNPQ has pairs of opposite sides equal, so it is a parallelogram.

Q3. Let M be the set of real numbers of the form $(m + n)/\sqrt{m^2 + n^2}$, where m and n are positive integers. Prove that for every pair x,y in M with $x < y$, there exists a z in M such that $x < z < y$.



Solution. There are many ways to solve this problem. The formula defining M may remind you of the cosine rule. Applying the cosine rule to the triangle POI in the figure gives

$$\cos\theta = \frac{a + b}{\sqrt{2}\,\sqrt{a^2 + b^2}}.$$

Alternatively,

$$\begin{aligned}\cos\theta &= \cos(\angle POx - 45^\circ)\\ &= \cos(\angle POx)\cos 45^\circ + \sin(\angle POx)\sin 45^\circ\\ &= \frac{b}{\sqrt{a^2 + b^2}}\cdot\frac{1}{\sqrt{2}} + \frac{a}{\sqrt{a^2 + b^2}}\cdot\frac{1}{\sqrt{2}}.\end{aligned}$$

(continued over)

Now let $x = (a + b)/\sqrt{a^2 + b^2}$ and $y = (c + d)/\sqrt{c^2 + d^2}$ be two numbers in M with $x < y$. We can assume $a \leqslant b$ and $c \leqslant d$, so the corresponding points fall as in the diagram with $\cos \angle POI < \cos \angle QOI$, that is $\angle POI > \angle QOI$. If $R = (e,f)$ is any point with integer coordinates in the sector formed by producing OP and OQ, then $\cos \angle ROI$ lies between $\cos \angle POI$ and $\cos \angle QOI$, whence $(e + f)/\sqrt{e^2 + f^2}$ lies between x and y, as required. One possible choice for R is the point $(a + c, b + d)$; in this case OPRQ is a parallelogram.

Q4. Let $f(x) = x^3 + ax^2 + bx + c$ be a cubic polynomial with integer coefficients a, b and c. Suppose that one root of the equation $f(x) = 0$ is equal to the product of the other two roots. Show that $2f(-1)$ is an integer multiple of $f(1) + f(-1) - 2f(0) - 2$.

Solution. Let the roots of the cubic be α, β and γ and suppose $\gamma = \alpha\beta$. From the formulae relating the roots and the coefficients of the cubic,

$$-a = \alpha + \beta + \gamma = \alpha + \beta + \alpha\beta$$

$$b = \alpha\beta + \beta\gamma + \gamma\alpha = \alpha\beta + \alpha\beta^2 + \alpha^2\beta$$

$$-c = \alpha\beta\gamma = \alpha^2\beta^2.$$

Thus

$$f(1) + f(-1) - 2f(0) - 2 = 2(a - 1) = -2(1 + \alpha + \beta + \alpha\beta)$$

and

$$2f(-1) = 2(-1 + a - b + c) = -2(1 + \alpha + \beta + \alpha\beta + \alpha\beta + \alpha\beta^2 + \alpha^2\beta + \alpha^2\beta^2)$$

$$= -2(1 + \alpha + \beta + \alpha\beta)(1 + \alpha\beta).$$

If we can show that $1 + \alpha\beta$ is an integer, then we will have shown that $2f(-1)$ is an integer multiple of $f(1) + f(-1) - 2f(0) - 2$, as required. Now, from the above equations,

$$1 + \alpha\beta = \frac{2(-1 + a - b + c)}{2(a - 1)} = 1 + \frac{c - b}{a - 1}, \text{ that is } \alpha\beta = \frac{c - b}{a - 1}.$$

This means that $\alpha\beta$ is a rational number, say $\alpha\beta = p/q$ in lowest terms. On the other hand, $\alpha^2\beta^2 = -c$, so $p^2 = -cq^2$. But this implies that q^2 divides p^2, whence q divides p, so that q must be equal to 1 since p/q was in its lowest terms. So, in fact, $\alpha\beta = p/1$ is an integer and so is $1 + \alpha\beta$. This completes the proof.

Q5. The function $f_d(x,y,z)$ where d is a positive integer, is the sum of all the distinct products $x^a y^b z^c$ where a,b,c are positive integers with $a + b + c = d$. For example, $f_1(x,y,z) = x + y + z$, and

$$f_2(x,y,z) = x^2 + y^2 + z^2 + yz + xz + xy.$$

If $x = 1$ and $y > 0$, $z > 0$ show that

$$f_{d+1}(x,y,z) > f_d(x,y,z).$$

If $x = 1$, $y = 1/3$ and $z = 2/5$ show that $f_d(x,w,z) < 2\frac{1}{2}$ for all d.

<u>Solution</u>. Suppose $x = 1$, $y > 0$ and $z > 0$. Then every term of $f_{d+1}(x,y,z)$ is positive. If $x^a y^b z^c$ is a term of $f_d(x,y,z)$, then $x^{a+1} y^b z^c$ is a term of $f_{d+1}(x,y,z)$. There are also other terms such as y^{d+1}. So

$$f_{d+1}(x,y,z) > \sum_{a+b+c=d} x^{a+1} y^b z^c = x f_d(x,y,z) = f_d(x,y,z)$$

Now suppose $x = 1$, $y = 1/3$ and $z = 2/5$. Since $x = 1$,

$$f_d(x,y,z) = \sum_{a+b+c=d} x^a y^b z^c = \sum_{b+c \le d} y^b z^c$$

$$< \sum_{b,c=0}^{\infty} y^b z^c = \left(\sum_{b=0}^{\infty} y^b\right)\left(\sum_{c=0}^{\infty} z^c\right)$$

$$= \frac{1}{1-y} \cdot \frac{1}{1-z}$$

$$= \frac{5}{2}.$$

Q6. We are given a function f with the property that, for all real numbers x,y

$$f(x + y) = f(x) + f(y).$$

Given $c \neq 0$ show how to choose M so that $g(x) = f(x) - Mx$ has period c. (This means $g(x + c) = g(x)$ for all x.) Prove

(i) if f is bounded then f is identically zero

(ii) if f is bounded on some interval then $f(x) = Mx$ for some M.

Note that a function is bounded if there is a constant k with $-k < f(x) < k$ for all x.

<u>Solution</u>. From the functional equation for f,

$$g(x + c) = f(x + c) - M(x + c) = f(x) + f(c) - Mx - Mc = g(x) + f(c) - Mc.$$

So we can arrange for $g(x + c) = g(x)$ by choosing $M = f(c)/c$.

(i) Suppose f is bounded, but $f(a)$, say, is non-zero. Then $f(2a) = f(a + a) = 2f(a)$, $f(3a) = f(2a + a) = f(2a) + f(a) = 3f(a)$, and, in general, $f(na) = nf(a)$ for any integer n. Since $f(a) \neq 0$, we can make $f(na) = nf(a)$ as large as we like by choosing a sufficiently large integer n. This contradicts the assumption that $f(x)$ is bounded. So f must be identically zero.

(ii) Now suppose $f(x)$ is bounded on the interval $a \leqslant x \leqslant b$. Construct $g(x) = f(x) - Mx$ to have period $b - a$, and note that $g(x)$ is also bounded on the interval $a \leqslant x \leqslant b$. Now $g(x)$ satisfies the functional equation $g(x + y) = g(x) + g(y)$ and $g(x)$ is bounded for all x because it is periodic and it is bounded over the period $a \leqslant x \leqslant b$. So, by (i), $g(x)$ is identically zero, and thus $f(x) = Mx$.

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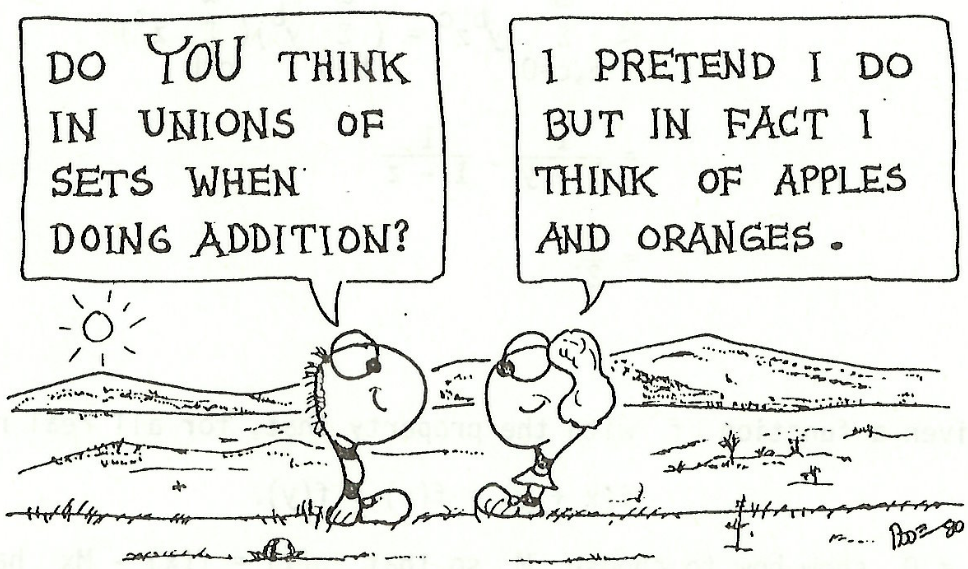

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