SOLUTIONS TO PROBLEMS FROM VOLUME 18, NUMBER 3

Q. 539. From the set of whole numbers $\{0, 1, 2, \ldots, 999999999\}$ two are selected at random. What is the probability that they differ by a multiple of 10000?

Solution. However the first number is chosen it can be expressed in the form $10000a + b$ where a, b are non-negative integers with $0 \leq b < 10000$. The second choice differs from this by a multiple of 10000 if it is given by $10000c + b$ where c is any non negative integer less than 10^5 except a. That is, of the $10^9 - 1$ numbers remaining after the first choice, there are $10^5 - 1$ which differ from it by a multiple of 10000. Thus the required probability is $\frac{10^5 - 1}{10^9 - 1}$. (This is very close to $\frac{1}{10000}$ which would be the exact answer if one interprets the question to allow the same number to be selected for both choices.)

Q. 540. Three married couples went shopping. Each of the six bought several articles and paid for each article a number of cents equal to the number he or she bought. (e.g. 7 articles each costing 7 cents would be a possibility.) Each women spent 75 cents more than her husband. Cecil spent 21 cents more than Mary, but Leah spent \$14.19 more than albert. The other two names were Nora and Brad. Who was married to whom?

Solution. Suppose a wife buys x articles, and her husband y articles. Then $75 = x^2 - y^2 = (x - y)(x + y)$. There are three factorisations of 75 with the second factor larger viz 1×75, 3×25, and 5×15. Solving $x - y = 1$, $x + y = 75$ gives $x = 38$, $y = 37$. The other factorisations give similarly $x = 14$, $y = 11$ and $x = 10$, $y = 5$.

Hence the number of cents spent by the three couples were:-

$$38^2 = 1444, \qquad 37^2 = 1369,$$
$$14^2 = 196, \qquad 11^2 = 121,$$
$$10^2 = 100, \qquad 5^2 = 25.$$

(continued over)

From the given information it is clear that Cecil spent 121 cents and Mary 100 cents, and that Leah spent 1444 cents and Albert 25 cents. Hence the couples were Leah and Brad, Nora and Cecil, Mary and Albert.

Q. 541. Andrew, Bud and Charles were seated one behind the other, Andrew at the back, Charles in front. From a bag containing 3 black hats and 2 white hats (all three were aware of its contents) one hat was placed on the head of each boy by a fourth person, who took care that no one saw the colour of the hat placed on his own head.

Andrew said "I cannot tell what colour hat is on my head".

Then Bud (who was not permitted to turn to look at Andrew) made the same statement. Would it be possible for Charles to deduce the colour of the hat on his head?

Solution. Obviously, if Andrew had been able to see 2 white hats he would have known that the hat he wore was black. So if Bud saw a white hat on Charles, he would have been able to argue that his hat must be black. Since he came to no such conclusion, Charles is able to deduce that his hat must be black (or else that Bud, though no doubt a worthy citizen in other respects, is solid concrete between the ears).

Q. 542. Suppose one's only drawing instrument is an ungraduated ruler with two straight parallel edges. One can use this instrument:

a) to draw a line through two given (or already constructed) points.

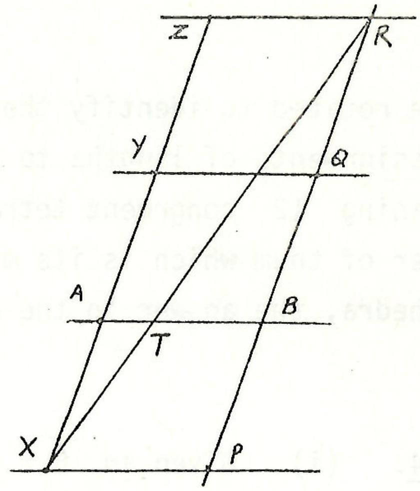
b) to draw a line parallel to a given line at a distance from it equal to the width of the ruler

c) to draw two parallel lines, one through each of two given points whose distance apart is at least equal to the width of the ruler.

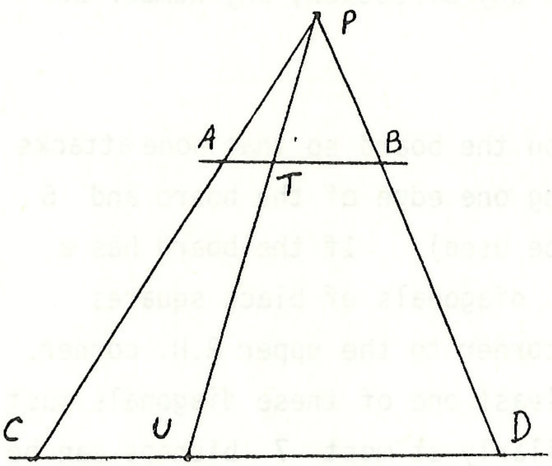
Show how to use this instrument to

1. trisect a line segment of length greater than the width of the ruler.
2. trisect a line segment of length less than the width of the ruler.

Solution. 1. Let AB be the given segment. Using operation c) construct parallel lines XZ and PR passing through A and B respectively. Using b) construct lines XP and QY parallel to AB, then ZR parallel to YQ. Join RX intersecting AB at T. Then T is a point of trisection of AB.



(Proof. The parallellograms XPBA, ABQY, and YQRZ are obviously congruent rhombuses. The triangles ΔXAT and ΔXZR are similar, whence $\frac{AT}{ZR} = \frac{XA}{XZ} = \frac{1}{3}$. Hence $\frac{AT}{AB} = \frac{1}{3}$.)



2. Using b) draw a line CD parallel to AB. Either by using 1 with CD sufficiently far apart, or otherwise, construct three points on this line with U a point of trisection of CD. Construct lines CA and DB intersecting at P, then PU intersection AB at T. Then T is a point of trisection of AB.

(Proof. In the similar triangles ΔPAT and ΔPCU $\frac{AT}{CU} = \frac{PA}{PC}$. In the similar triangle ΔPAB and ΔPCD $\frac{AB}{CD} = \frac{PA}{PC}$. Thus $\frac{AT}{CU} = \frac{AB}{CD}$, whence $\frac{AT}{AB} = \frac{CU}{CD} = \frac{1}{3}$.)

Q. 543. Six different lengths are given, such that they can be taken in any order as the edges of a tetrahedron ABCD.

How many non-congruent tetrahedra can be constructed in this way? (Count "mirror image" tetrahedra as non-congruent.)

Solution. There are altogether $6! = 720$ ways of assigning the six given lengths to the edges AB, AC, AD, BC, CD, and BD.

Once one possible tetrahedron has been constructed, it can be rotated into 12 different positions when identifying its sides with those of ABCD (for each of the 4 vertices which can be identified with A, there are three positions into which it

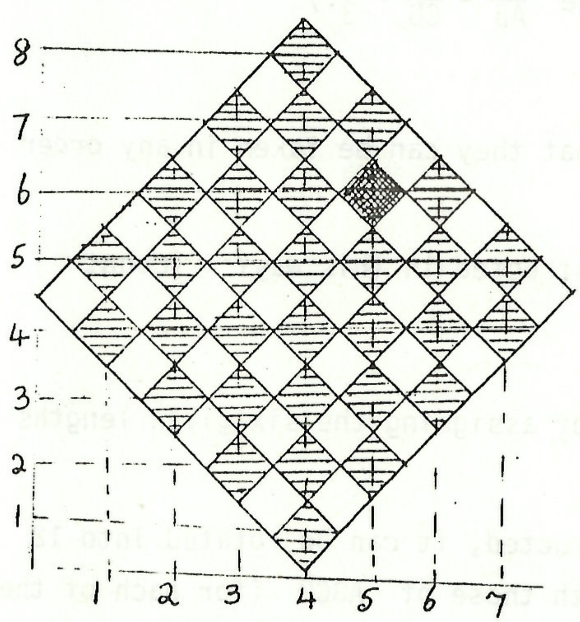
(continued over)

can be rotated to identify the opposite face with the triangle BCD.) Hence the 720 assignments of lengths to the sides of ABCD fall into 60 classes each containing 12 congruent tetrahedra. [For each of these 60 classes, there is another of them which is its mirror image; if these were regarded as congruent tetrahedra, the answer to the question would be 30 instead of 60.]

Q. 544. (i) Given an 8×8 chessboard, what is the largest number of bishops which can be placed on the board so that no bishop attacks any other?

(ii) In how many different ways can this maximum number of bishops be placed on the board? (A bishop moves diagonally in any direction, any number of squares.)

Solution. (i) One can place up to 14 bishops on the board so that none attacks another. One way to do this is by placing 8 along one edge of the board and 6 along the opposite edge (its corner squares cannot be used). If the board has a white square in the lower R.H. corner, there are 7 diagonals of black squares parallel to the long diagonal from the lower L.H. corner to the upper R.H. corner. If more than 7 bishops were on black squares, at least one of these diagonals must contain 2 bishops, which attack each other. Similarly at most 7 bishops can be placed on the white squares. This proves that 14 is the largest possible number.



(ii) We shall show that there are $2^4 = 16$ different ways of placing 7 non-attacking bishops on black squares, and similarly 16 ways of placing 7 bishops on white squares, yielding $16 \times 16 = 256$ different ways altogether of placing all 14 bishops.

Let the board be tilted through 45° with a black square at the bottom corner. There are now 7 vertical columns and 8 horizontal rows of black squares. A two co-ordinate label, such as (5,6), identifies the black square in the 5th vertical column counting from the left and the 6th horizontal row counting from the bottom, (on the

(continued over)

figure (5,6) is the solidly shaded square).

The seven squares for placing non-attacking bishops must have labels

$$(1,x_1),\ (2,x_2),\ (3,x_3),\ (4,x_4),\ (5,x_5),\ (6,x_6)\ \text{and}\ (7,x_7)$$

where no two of $x_1, x_2, \ldots, x_7$ are equal. Since x_1 is either 4 or 5 (there are only 2 squares in the first column) and the same is true for x_7, there are just 2 ways of choosing x_1 and x_7; viz either $x_1 = 4$, $x_7 = 5$ or $x_1 = 5$, $x_7 = 4$. That leaves only 3 and 6 as available choices for x_2 and x_6; again there are two ways of choosing the pair. Similarly x_3 and x_5 have now to be chosen either as $x_3 = 2$, $x_5 = 7$ or $x_3 = 7$, $x_5 = 2$. Finally x_4 must be chosen to be either 1 or 8 since all other row numbers have been used up. Hence, as asserted above, there are $2 \times 2 \times 2 \times 2$ ways of placing the 7 bishops on the black squares.

Q. 545. The squares on an $n \times n$ chessboard are to be coloured red, yellow, blue or green in such a way that squares with a common edge or corner are given different colours. Is it possible to do this in such a way that every row or column contains at least one square of each colour?

Solution. If in some row squares of each of the 4 colours occur, we can certainly find 3 consecutive squares in the row bearing different colours 1, 2, and 3 say. Now the square vertically above (or below) that coloured 2 must be coloured with the fourth colour, 4, and then the colours of its Left and Right neighbours are forced to be 3 and 1 respectively.

Now the three colours in the next row up (or down) are similarly determined to be from left to right ... 1, 2, 3, ...

...	...	...	...	...
...	1	2	3	...
...	3	4	1	...
...	1	2	3	...
...	...	...	...	...

i.e. identical with those of the first row. There is thus no choice in completing the colouring of these three columns - in the outer two the squares are coloured alternatively 1 and 3; in the middle one the squares are coloured alternatively 2 and 4.

(continued over)

Thus if any row contains all four colours it is impossible that the same is true for all the columns. In fact, no column can contain more than 2 colours, since once any column is so coloured, a neighbouring column can have only the other two colours.

Q. 546. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m,n,

$$f(m + n) - f(m) - f(n) = 0 \text{ or } 1;$$

$$f(2) = 0,\ f(3) > 0, \text{ and } f(9999) = 3333.$$

Determine $f(1982)$.

Solution. From $0 = f(2) = f(1) + f(1) + (0 \text{ or } 1)$ we deduce that $f(1) = 0$. From $0 < f(3) = f(1) + f(2) + (0 \text{ or } 1) = 0 + 0 + (0 \text{ or } 1)$ we deduce that $f(3) = 1$.

We now prove by induction that

$$\text{for every } k, \quad f(3k) \geqslant k. \tag{1}$$

This has already been established for $k = 1$. Assuming it is true when $k = k_0$, then

$$\begin{aligned} f(3(k_0 + 1)) &= f(3k_0) + f(3) + (0 \text{ or } 1) \\ &\geqslant k_0 + 1 + (0 \text{ or } 1) \\ &\geqslant k_0 + 1. \end{aligned}$$

Thus the assertion (1) remains true for $k = k_0 + 1$ if it is true at $k = k_0$, so it is true for all k.

The working in the proof of (1) shows further that if $f(3k_0) > k_0$ then $f(3(k_0 + 1)) > k_0 + 1$. It follows immediately that if $f(3\ell) = \ell$ for some value of ℓ, then $f(3k) = k$ for all values of k less then ℓ. In particular, since $f(3 \times 3333) = 3333$, we have $f(1980) = f(3 \times 660) = 660$,

$$\text{and also } f(3 \times 1982) = 1982. \tag{2}$$

$$\begin{aligned} \text{Now } f(1982) &= f(1980) + f(2) + (0 \text{ or } 1) \\ &= 660 + 0 + (0 \text{ or } 1) = 660 \text{ or } 661. \end{aligned}$$

It remains to determine which of these two possible answers is the correct one.

(continued over)

If $f(1982) = 661$ then $f(3964) = f(1982) + f(1982) + (0 \text{ or } 1)$

$$\geqslant 1322,$$

and $f(3964 + 1982) \geqslant f(3964) + f(1982) \geqslant 1322 + 661 = 1983$ i.e. $f(3 \times 1982) > 1982$. Since this contradicts (2) it is impossible that $f(1982) = 661$. We must have $f(1982) = 660$.

Q. 547. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1 \text{ and for all } i \geqslant 0, \quad x_{i+1} \leqslant x_i.$$

a) Prove that for every such sequence, there is an $n \geqslant 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \dots + \frac{x_{n-1}^2}{x_n} \geqslant 3.999 \, .$$

b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \dots + \frac{x_{n-1}^2}{x_n} < 4 \quad \text{for all} \quad n.$$

Solution. a) We shall refer to a sequence $\{a_n\}$ with the given properties as a DP1 sequence (decreasing, positive, first term = 1), and the expression $\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \dots + \frac{x_{n-1}^2}{x_n}$ will be denoted by E_n.

Note that $E_2 \geqslant \frac{1}{x_1} + x_1$ (since $x_0 = 1$ and $\frac{x_1}{x_2} \geqslant 1$)

$$\geqslant \frac{(x_1 - 1)^2}{x_1} + 2 \geqslant 2.$$

and that $E_n > E_{n-1} > E_{n-2} > \dots > E_2 \geqslant 2$ for every $n > 2$ and every DP1 sequence. (1)

If $\{x_n\}$ is a DP1 sequence, so is $\{x_n'\}$ where $x_n' = \frac{x_{n+1}}{x_1}$ for $n = 0,1,2,\dots$.

We let E_n' denote $\frac{x_0'^2}{x_1'} + \frac{x_1'^2}{x_2'} + \dots + \frac{x_{n-1}'^2}{x_n'}$.

(continued over)

Observe that $E_n' = \frac{\left(\frac{x_1}{x_1}\right)^2}{\left(\frac{x_2}{x_1}\right)} + \frac{\left(\frac{x_2}{x_1}\right)^2}{\left(\frac{x_3}{x_1}\right)} + \ldots + \frac{\left(\frac{x_n}{x_1}\right)^2}{\left(\frac{x_{n+1}}{x_1}\right)}$

$$= \frac{1}{x_1}\left[\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ldots + \frac{x_n^2}{x_{n+1}}\right] = \frac{1}{x_1}\left[E_{n+1} - \frac{1}{x_1}\right] = \frac{E_{n+1}^2}{4} - \frac{1}{4}(E_{n+1} - \frac{2}{x_1})^2$$

This yields $E_n' \leqslant \frac{E_{n+1}^2}{4}$ for every n. Hence if $\{x_n\}$ is a DP1 sequence such that every $E_{n+1} \leqslant k$, then $\{x_n'\}$ is another DP1 sequence with every $E_n' \leqslant \frac{k^2}{4}$. Repeating the process we could then find yet another DP1 sequence $\left(\{x_n''\} \text{ where } x_n'' = \frac{x_{n+1}'}{x_1'} = \frac{x_{n+2}}{x_2}\right)$ for which every $E_n'' \leqslant \frac{\left(\frac{k^2}{4}\right)^2}{4} = \frac{k^{2^2}}{4^{2^2-1}}$; and in fact after r repetitions we would have a DP1 sequence

$$\left(\{x_n^{(r)}\} \text{ where } x_n^{(r)} = \frac{x_{n+1}^{(r-1)}}{x_1^{(r-1)}} = \frac{x_{n+k}}{x_r}\right)$$

with the property that every $E_n^{(r)} \leqslant \left(\frac{k}{4}\right)^{2^r-1} k$ (Easy check by induction on r).

If $k = 3.999$, we have $(\frac{k}{4}) < 1$ so that for a large enough value of r we have constructed a DP1 sequence for which $E_n < \frac{1}{2} . 3.999 < 2$ for every n.

Since this contradicts (1), it follows that there exists no DP1 sequence for which $E_n \leqslant 3.999$ for every n. Q.E.D.

b) If $E_n < 4$ for every n, then $E_n' \leqslant \frac{E_{n-1}^2}{4} < \frac{4^2}{4} = 4$ for every n. The working of (a) shows that $\lim_{n\to\infty} E_n = 4$ and similarly $\lim_{n\to\infty} E_n' = 4$.

Letting n tend to infinity in $E_n' = \frac{1}{x_1}\left[E_{n+1} - \frac{1}{x_1}\right]$ yields $4 = \frac{1}{x_1}\left[4 - \frac{1}{x_1}\right]$

(continued over)

which solves to give $x_1 = \frac{1}{2}$. Similarly $x_1' = \frac{1}{2}$, giving $x_2 = x_1' \cdot x_1 = \frac{1}{2^2}$.

We could continue this; but it is simpler to merely make the obvious guess that the sequence $x_n = \frac{1}{2^n}$ $n = 0, 1, 2, \ldots$ is the one we are looking for, without proving that there is no other possibility. For this DP1 sequence we have

$$E_n = \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2^k}\right)^2}{\frac{1}{2^{k+1}}} = \sum_{k=0}^{n-1} \frac{2}{2^k} = \frac{2\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} \quad \text{(by the formula for summing a G.P.) i.e.}$$

$E_n < 4$ for every n.

Q. 548. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x,y), then it has at least three such solutions. Show that the equation has no solution in integers when $n = 2891$.

Solution. Note that

$$\begin{aligned} x^3 - 3xy^2 + y^3 &= (y - x)^3 - 3x^2y + 2x^3 = (y - x)^3 - 3x^2(y - x) - x^3 \\ &= X^3 - 3XY^2 + Y^3 \end{aligned}$$

$$\text{where } X = y - x, \quad Y = -x \tag{1}$$

Hence if $(x,y) = (x_1,y_1)$ is one solution in integers of the equation, another solution is $(x_2, y_2) = (y_1 - x_1, -x_1)$. The same transformation performed on (x_2,y_2) gives yet another solution $(x_3,y_3) = (y_2 - x_2, -x_2) = (-y_1, x_1 - y_1)$. One can check immediately that performing the transformation (1) on (x_3,y_3) returns us to the original solution (x_1,y_1).

It can be verified easily that when the transformation (1) is performed it is not true that $X = x$, $Y = y$ except when $x = y = 0$. It follows that the three solutions when n is a positive integer are all different.

Observe that $2981 = 7^2 \times 59$. Note first that:- There is no solution of

(continued over)

$x^3 - 3xy^2 + y^3 = 2891$ for which either x or y is a multiple of 7. (2)

(Proof. Suppose $7|x$. Then $y^3 = 2891 - x^3 + 3xy^2$, a multiple of 7. Hence $7|y$. But then 7^3 is a factor of $x^3 - 3xy^2 + y^3$, but not of 2891, a contradiction.)

One readily checks that $(7k + r)^3 = 7(49k^3 + 21k^2r + 3kr^2) + r^3$ leaves the same remainder on division by 7 as r^3. When $r = 1, 2, 3, 4, 5,$ or 6 the remainder when r^3 is divided by 7 is 1, 1, 6, 1, 6, and 6 respectively.

Now suppose there exists a solution $(x,y) = (x_1,y_1)$ of $x^3 - 3xy^2 + y^3 = 2891$. Then, of the three solutions $(x,y) = (x_1,y_1)$, $(x,y) = (y_1 - x_1, - x_1)$, $(x,y) = (- y_1\ x_1 - y_1)$ there must always be at least one for which one of x^3,y^3 leaves remainder 1, the other remainder 6, on division by 7.

(Proof. If x_1^3 and y_1^3 both leave the same remainder the same is true of $(- x_1)^3$ and $(- y_1)^3$. But then since $(x_1 - y_1)^3$ and $(y_1 - x_1)^3$ must leave remainders 1 and 6, the assertion must be true for either the second or third pair).

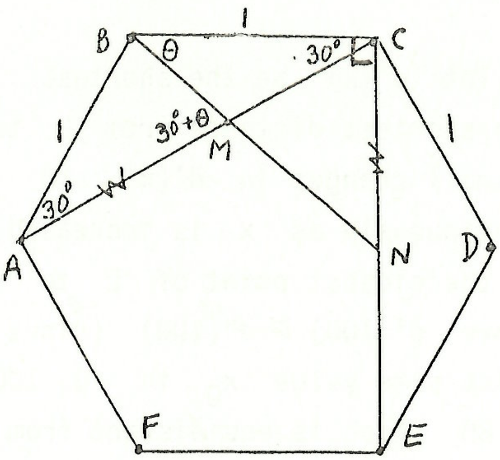
For such a solution (x,y) of the equation we have $x^3 + y^3$ exceeds a multiple of 7 by $1 + 6$, i.e. it is a multiple of 7. Then $3xy^2 = x^3 + y^3 - 2891$ is also a multiple of 7. But this is impossible, since we have already shown that neither x nor y can be a multiple of 7.

Because of this contradiction, we can deduce that no solution exists.

Q. 549. The diagonals AC and CE of the regular hexagon ABCDEF are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear.



<u>Solution</u>. Let the side length be 1. Then $AC = CE = 2 \cos 30^\circ = \sqrt{3}$; $\angle BCE = 90$; $AM = CN = \sqrt{3}\, r$.

Let $\angle CBN = \theta$. Then $\angle BMA = \theta + 30^\circ$; $\angle ABM = 120^\circ - \theta$; and $CN = BC \tan\theta = 1 \tan\theta$;

The sine rule for ΔABM gives

$$\frac{AM}{\sin \angle ABM} = \frac{AB}{\sin \angle ABM}$$

$$\frac{\tan\theta}{\sin(120^\circ - \theta)} = \frac{1}{\sin(\theta + 30^\circ)}$$

$$\sin\theta(\sin\theta \cos 30^\circ + \cos\theta \sin 30^\circ) = \cos\theta(\sin 120^\circ \cos\theta - \cos 120^\circ \sin\theta)$$

$$\frac{\sqrt{3}}{2}\sin^2\theta + \frac{1}{2}\sin\theta\cos\theta = \frac{\sqrt{3}}{2}\cos^2\theta + \frac{1}{2}\sin\theta\cos\theta.$$

$\therefore \sin^2\theta = \cos^2\theta$. As θ is acute, we have $\theta = 45^\circ$, and $AM = CN = 1 \tan 45^\circ = 1$.

$\therefore r = \frac{1}{\sqrt{3}} \approx .577.$

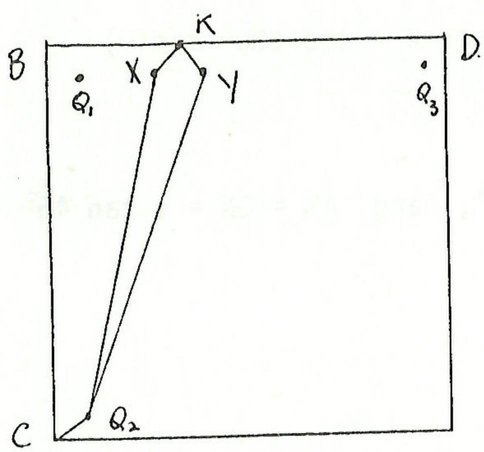
<u>Q. 550</u>. Let S be a square with sides of length 100 and let L be a path within S which does not meet itself and which is composed of linear segments $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than 1/2. Prove that there are two points X and Y in L such that the distance between X and Y is not smaller than 198.

<u>Solution</u>. Traverse L from A_0 to A_n. Let Q_1 be the first point reached such that the distance from Q_1 to a corner, B, of the square is $\leqslant$ the distance from B to any other point of L (i.e. Q_1 is a "closest point" of L to B). Similarly let Q_2 be the next point of L reached which is a "closest point" of L to a different corner, C, of the square. Let D be a third corner of the square such that BD is an edge; let Q_3 be a (the) nearest point of L to D. Denote by L' the portion $A_0 - Q_2$ of L, and by L'' the remaining portion $Q_2 - A_n$; note that

(continued over)

Q_3 is a point of L''.

If P is any point on BD, x units from B, let $d'(x)$ be the shortest distance from P to a point of L', and $d''(x)$ the shortest distance from P to a point of L''. A small change in x causes only small changes in $d'(x)$ and $d''(x)$, obviously, i.e. $d'(x)$ and $d''(x)$ vary continuously as x is increased from 0 to 100. At $x = 0$ $d'(0) \leqslant d''(0)$ (since the closest point of L to B is Q_1 which lies on L'); and at $x = 100$ we have $d'(100) \geqslant d''(100)$ (since Q_3 lies on L''). It is evident that there must exist some value x_0 in $[0, 100]$ for which $d'(x_0) = d''(x_0)$. i.e. some point K in BD which is equidistant from L' and L''. (It is not necessary to prove that there is only one such point K, though this is true because L does not intersect itself.)



Let X,Y be the points of L' and L'' respectively which are closest to K. then $KX = KY \leqslant \frac{1}{2}$ and $XY < KX + KY < 1$. The portion of L between X and Y passes through Q_2 so its length is not less than $XQ_2 + YQ_2$.

Since

$$KX + XQ_2 + 2Q_2C + Q_2Y + YK \geqslant 2KC \geqslant 2BC = 200$$

with KX, KY and Q_2C all $\leqslant \frac{1}{2}$ we have $XQ_2 + YQ_2 \geqslant 200 - 4 \times \frac{1}{2} = 198$, and the proof is complete.

PROBLEM SOLVERS

Teresa Keo (Presbyterian Ladies College Croydon) 541. Tim Bedding (Beacon Hill High School) 541, 544, 545, (The arguments for these solutions were identical with those published above). Rolfe Bozier (Barker College) 541, 542, 546*, 547*, 549. (*Although the arguments for the difficult Olympiad problems 546 and 547 were judged to be incomplete, the contributions were still quite meritorious). A. Jenkins (North Sydney Boys High School) 541, 542 (excellent solutions)