THE 1983 SCHOOL MATHEMATICS COMPETITION*

SOME COMMENTS ON THE EXAMINATION

JUNIOR DIVISION

In the Junior Section of the competition there were three questions which were fairly easy, two rather difficult, and one impossibly so.

The easy questions were question 1, which was a set of equations, question 4, which concerned the various sports played or not played by Jim Adams, and question 5, which concerned the moves of a duke.

Most prizewinners solved all these three problems, apart from showing that 5 moves is the minimum in the chess problem.

A large number of people did not understand what was meant in question 2 by a "3 × 3 table", which is unfortunate, because they may have made some progress if they had understood. Some saw that there are only seven possible row (or column) totals, namely -3, -2, ..., 3, and scored a mark for that. Some could show that if all six in the table are supposed different, then 3 and -3 cannot both occur, and they were rewarded with half-marks for the question. Perhaps twenty candidates completed the solution, getting full marks.

A small number of people managed to solve question 3, the problem concerning the operation # , but only a handful showed their solution was the only correct solution and scored full marks.

In the last question, which was admittedly a tricky one, a few candidates came close to discovering the main idea but failed to carry it through, probably because they did not think of giving the lowest common multiple a name, say M.

After the exam, Meredith Jordan of Monaro High School found a very nice way of solving the problem: Let M be the least common multiple of the numbers $a_1, \dots, a_n$. Then there are positive integers $k_1, \dots, k_n$ such that

$$k_1a_1 = k_2a_2 = \dots = k_na_n = M.$$

Since $a_1 < a_2 < \dots < a_n$, we must have $k_1 > k_2 > \dots > k_n$. Thus $k_1 \geqslant n$ and $M \geqslant na_1$, as required. This was one of the rare occasions on which there was a problem for which no-one scored full marks.

M.D. Hirschhorn.

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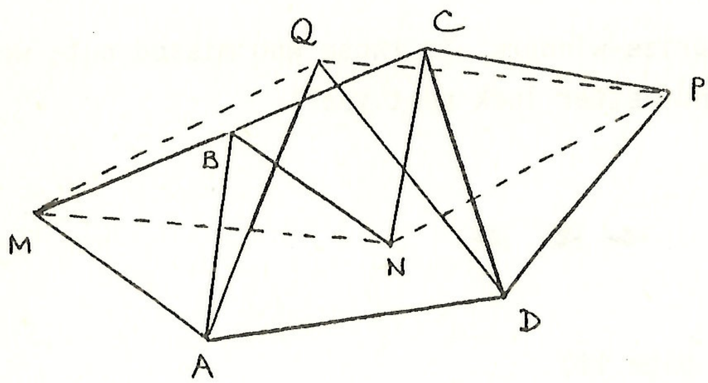
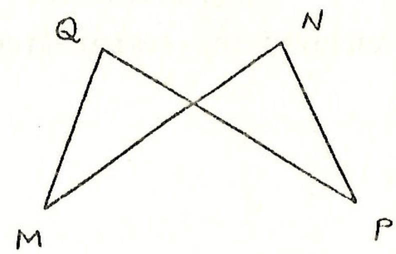
The first two questions were comparatively well done, with little variation from the methods given previously. Dr. M. Newman (A.N.U.) has pointed out an elegant approach to question 2. Let R_1 denote a clockwise rotation of 60° about A and let R_2 denote an anticlockwise rotation of 60° about C. Then $T = R_2R_1$, that is R_1 followed by R_2, is a translation and, from the figure, T sends M to N and Q to P. Thus T sends MN to QP and MNPQ is a parallelogram. Note that the original quadrilateral ABCD need not be convex. The solution given in the last issue is not quite complete. A quadrilateral MNPQ with opposite side equal need not be a parallelogram. It could be crossed, as shown, and this possibility should be eliminated.





For question 3, few candidates recognized that M was the set of numbers attained by the function $\dfrac{1 + x}{\sqrt{1 + x^2}}$ when x is a national number ($\frac{m}{n}$ or $\frac{n}{m}$) in the range $0 < x \leqslant 1$. From a more advanced standpoint than was, or could be, expected from those taking the examination the desired result follows immediately from standard properties of continuous functions, together with the "denseness" of the rationals in the real number continuum. A few very good students found trigonometric implications such as $\dfrac{1 + x}{\sqrt{1 + x^2}} = \sqrt{1 + \sin\theta}$ when $x = \tan\theta$. Another successful method was to show that $\dfrac{km + (kn+1)}{\sqrt{(km)^2 + (kn+1)^2}}$ could be made as close as desired to $\dfrac{m + n}{\sqrt{m^2 + n^2}}$ by taking k to be a sufficiently large positive integer.

In question 4, quite a number of students got as far as showing that the ratio of the two given (integer) quantities is $1 + \alpha$ where α is the root equal to the product of the other two. Very few continued by observing that this fact forced α to be a rational number whose square is the integer $-c$; hence α must be an integer.

In answering question 5 a few students commented on the inaccuracy of the statement of the equation, where "positive integer" should have read "non negative integer A large proportion of attempts foundered in an ocean of inconclusive algebraic manipulation.

Only a few students made any progress with question 6 and no-one gave a completely satisfactory argument for the last part of the question.

The first prize-winners were awarded marks equivalent to 5 questions of the 6 correct, a very commendable performance indeed. The equivalent of about 2½ questions correct was needed to get into the prize list.

C.D. Cox.

Congratulations to all the prize-winners. To those who missed out, we hope you enjoyed the competition anyway, and better luck next year!

OLYMPIAD OMNIBUS (Continued from page 11)

Problem 0.0.1 and solution

This problem requires the step of seeking integer solutions to a general problem.

> A rectangular floor, whose length and breadth are whole numbers of feet, is tiled. Each tile is one foot square. Find all possible dimensions for the floor such that the number of tiles around the perimeter is exactly half the total number of tiles.

Solution: Let the sides be of length a ft, b ft respectively.

Then $\frac{1}{2}ab = 2a + 2b - 4$.

(Check this step)

$$\therefore \quad b = \frac{4a - 8}{a - 4} = 4 + \frac{8}{a-4}.$$

This is an integer if $a-4$ divides 8. Hence the dimensions are 5×12 or 6×8.

[See page 18 for problem 0.0.2.]