PROBLEM SECTION

As usual, I have included five of the problems from this years International Olympiad (problems 583 - 587). They are of course fiendishly difficult - if you can find a perfect solution of even only one of them in a couple of hours you are in world class as a school student mathematician. Problem Q576 is modelled after a similar problem in "The Mathematics Student", a now discontinued American school mathematics journal.

Q576. The game of Yellow Pigs is a favourite pastime at Hampshire College's Summer Science Training Program. It is played as follows. One player chooses three different numbers from the first seven positive integers to form a three digit number, and a second player tries to guess the chosen number by testing different three digit numbers in succession. Each guess is given an answer in the form "k pigs, n of them are yellow" meaning that k of his three digits were correct, and of these, n occupied the same position as in the chosen number. In one such game each of the first four guesses; 631, 237, 253, and 425 were answered by "1 pig; it is not yellow". What was the chosen number?

Q577. ABC is a triangle of area 1 unit. Points X,Y,Z lie on BC (produced), CA (produced) and AB (produced), so that BX = 3 times BC, CY = 3 times CA, and AZ = 3 times AB. Find the area of Δ XYZ,

Q578. I have two perfect squares whose product exceeds their sum by 4844. Find them.

Q579. A set A contains n distinct numbers. The set S is constructed from A by the definition $S = \{x + y : x,y \in A\}$. i.e. the numbers in S are obtained by adding two numbers in A. (Note that x and y may be the same element of A.) Let m be the number of elements in the set S. Find the smallest and the largest possible values of m.

Q580. The lengths of the sides of a triangle ABC are all rational numbers. Let D be the foot of the perpendicular from A to BC. Show that the length of BD is a rational number.

Q581. The symbol $[x]$ denotes the greatest integer less than or equal to x. Find the complement of the set $E = \{[n + \sqrt{n + \frac{1}{2}}] : n \in \mathbb{N}\}$ in the set $\mathbb{N}$ of natural numbers. (i.e. Describe all whole numbers m such that m is not equal to $[n + \sqrt{n + \frac{1}{2}}]$ for any whole number n.)

Q582. In the arithmetic progression

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \ldots, a + nd, \ldots$$

three consecutive terms are perfect squares. Prove that d is a multiple of 24.

Q583. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the condidtions:

i) $f(xf(y)) = yf(x)$ for all positive x, y

ii) $f(x) \to 0$ as $x \to +\infty$.

Q584. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centres O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that the angles O_1AO_2 and M_1AM_2 are equal.

Q585. Let a, b, c be positive integers, no two of which have a common divisor greater than 1. Show that

$$2abc - ab - bc - ca$$

is the largest integer which cannot be expressed in the form

$$xbc + yca + zab$$

where x, y, z are non-negative integers.

Q586. Let ABC be an equilateral triangle, and E be the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

Q587. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithemtic progression? Justify your answer.

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