SOLUTIONS TO PROBLEMS FROM VOLUME 19 NO.1

Q.551. Solve the system of equations

$$u^2 + v^2 + w^2 = 49 \qquad (1)$$

$$\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 0 \qquad (2)$$

$$u - v + w = 1 \qquad (3)$$

Answer Multiplying (2) by uvw yields

$$vw + uw + uv = 0 \qquad (4)$$

Since $(u + v + w)^2 = u^2 + v^2 + w^2 + 2(vw + uw + uv) = 49 + 2 \times 0 = 49$ we obtain

$$u + v + w = \pm 7 \qquad (5)$$

Adding, and substracting, equations (3) and (5) yields

$$\begin{cases} u + w = \frac{8}{2} \text{ or } u + w = -\frac{6}{2} & (6) \\ v = 3 \qquad v = -4 & (7) \end{cases}$$

From $1 = (u - v + w)^2 = u^2 + v^2 + w^2 - 2(vw + uw + wv) + 4uw = 49 + 4uw$ it follows that $uw = -12$ (8)

Eliminating u between (6) and (8) gives

$$w^2 - 4w - 12 = 0 \quad \text{or} \quad w^2 + 3w - 12 = 0.$$

$$w = 6 \text{ or } -2 \qquad w = \frac{-3 \pm \sqrt{57}}{2}$$

$$u = -2 \text{ or } 6 \qquad u = \frac{-3 \mp \sqrt{57}}{2}.$$

Thus there are four solutions $(u,v,w) = (-2, 3, 6)$; $(6,3,-2)$;

$$\left(\frac{-3-\sqrt{57}}{2}, -4, \frac{-3+\sqrt{57}}{2}\right); \text{or} \left(\frac{-3+\sqrt{57}}{2}, -4, \frac{-3-\sqrt{57}}{2}\right).$$

Correct solutions from: G. Low (Sydney Technical H.S.); J. Mok (P.L.C. Croydon); J. Percival (Elderslie H.S.).

Slightly incomplete solutions were received from J. Hardy (Darmalan College, A.C.T.) and P. van Nguyen (Fairfield H.S.).

Q.552. Prove that if x,y and z are all positive numbers then

(a) $xy(x+y) + yz(y+z) + zx(z+x) \geqslant 6xyz$

and (b) $x^3 + y^3 + z^3 \geqslant 3xyz$.

Answer (a) Since $z > 0$ and $(x-y)^2 \geqslant 0$, $z(x-y)^2 \geqslant 0$. Adding this and two similar inequalities gives

$$z(x-y)^2 + y(z-x)^2 + x(y-z)^2 \geqslant 0$$

$$zx^2 + zy^2 + yz^2 + yx^2 + xy^2 + xz^2 - 6xyz \geqslant 0$$

$$xy(x+y) + yz(y+z) + zx(z+x) \geqslant 6xyz . \qquad \text{Q.E.D}$$

(b) Since $(x-y)^2 \geqslant 0$, $x^2 + y^2 \geqslant 2xy$. Adding this and two similar inequalities $(x^2+y^2) + (y^2+z^2) + (z^2+x^2) \geqslant 2(xy+yz+zx)$ whence $x^2 + y^2 + z^2 - xy - yz - zx \geqslant 0$.

Since $x+y+z > 0$, $(x+y+z)(x^2+y^2+z^2 - xy - yz - zx) \geqslant 0$

$$x^3 + y^3 + z^3 - 3xyz \geqslant 0 . \qquad \text{Q.E.D.}$$

The above argument was supplied by J. Percival. An equally good answer was received from G. Low.

Q.553. In a hat are six cards identical except for their colouring. Each side of every card is red, or white, or blue and no two cards are identically coloured. (So the cards are coloured RR, RW, RB, WW, WB and BB). One card is drawn from the hat and placed on a table. The visible face is red. Find the probability that the hidden face is also red.

Answer J. Hardy (Darmalan College, A.C.T.) writes:

There are four faces in all coloured red. Of these four, two have the same colour (red) on the hidden side. Hence the probability is 1 in 2 that the concealed face is also red.

(Several other contributors believed incorrectly that the answer to the problem is $\frac{1}{3}$.)

Q.554. Find three infinite sets A, B, C of non negative integers such that every non-negative integer can be written uniquely as the sum of an element from A, an element from B, and an element from C.

Answer There are infinitely many different solutions.

One is the following: Let A consist of all whole numbers whose ordinary decimal expression has the digit 0 in the "10^n" column for $n = 1,2,4,5,7,8,\ldots\ldots,$ $3k+1$, $3k+2,\ldots\ldots$. For example, $a = 5004000001003$ would be placed in A, since the only non-zero digits of this number lie in the columns for 10^n where $n = 0,3,9$ and 12. Similarly, let B contain all whole numbers whose non-zero digits are all in the "10^n" columns where $n = 3k + 1$, $k = 0,1,2,3,\ldots$; (e.g. f = 20010 is in B.). Finally C contains all numbers whose non-zero digits are in the columns for 10^n, where $n = 3k+2$, $k = 0,1,2,\ldots\ldots$ (e.g. c = 600700 is in C.) It is clear that any given integer (e.g. x = 5004000621713) can be uniquely expressed as the sum of numbers from A, B & C. (Our x is the sum of the numbers a, b, and c above).

Q.555. The ratio of the speeds of two trains is equal to the ratio of the times they take to pass each other going in the same or in opposite directions on parallel tracks. Find that ratio.

Answer If v_1 and v_2 are the speeds of the two trains, $v_1 > v_2$, the times taken to pass are $\frac{d}{v_1+v_2}$ and $\frac{d}{v_1-v_2}$, where d is the combined lengths of the trains.

We are given $\frac{d}{v_1-v_2} : \frac{d}{v_1+v_2} = v_1 : v_2$

i.e. $\frac{v_1+v_2}{v_1-v_2} = \frac{v_1}{v_2}$. Setting $x = \frac{v_1}{v_2}$, we obtain

$\frac{x+1}{x-1} = x$, or $x^2 - 2x - 1 = 0$, yielding $x = +1 + \sqrt{2}$

after rejecting the negative root.

The above argument is a somewhat condensed version of that sent by J. Percival.

Q.556. The points of (3-dimensional) space are coloured with one of five colours, each colour being used for at least one point. Prove that in some plane at least 4 different colours occur.

Answer If any line ℓ contains points bearing three different colours, let P be a point coloured with a fourth colour. A plane through ℓ and P contains four colours (at least), and we are finished. So let us suppose that no line contains more than 2 colours. Consider the plane determined by points A, B, C coloured a, b, c respectively. All points on the line BC are coloured b or c.

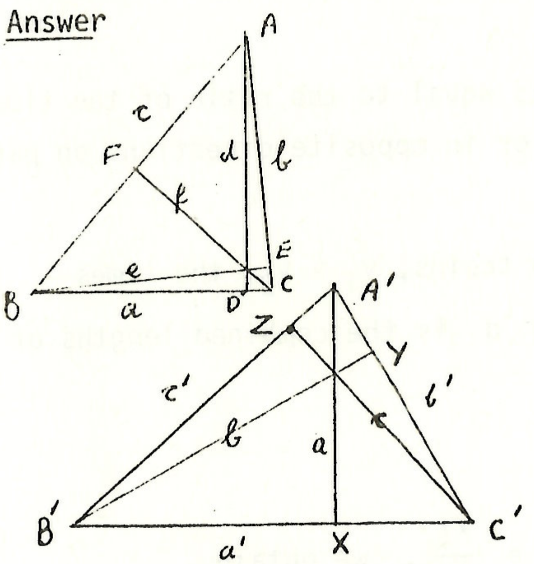
Now let points D, E bear the remaining colours, d, e. If the line DE pierces the plane ABC at P then P being on the line DE has colour d or e, and the plane ABCP has 4 colours.

However if DE is parallel to the plane ABC let AX be a parallel line in plane ABC. If AX intersects the line BC at Q then Q has colour b or c and the plane DEAQ has four colours.

If AX, (or DE), should happen to be parallel to BC then the plane DEBC has four colours.

Q.557. A triangle is given. Assuming it is possible, show how to construct with ruler and compass a second triangle whose three altitudes are equal in length to the sides of the given triangle.

Answer



In the diagram, ΔABC is the given triangle, and AD, BE and CF are its altitudes of lengths d, e and f respectively. $\Delta A'B'C'$ is the required triangle having altitudes A'X of length a (= length of BC), B'Y of length b, and C'Z of length c.

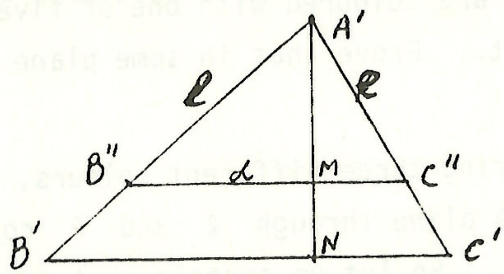
Let $k = \dfrac{\text{area of } \Delta A'B'C'}{\text{area of } \Delta ABC}$. Then since

$B'C' \times A'X = 2\times$ area of $\Delta A'B'C'$

$BC \times AD = 2 \times$ area of ΔABC we obtain

$\dfrac{a' \cdot a}{a \cdot d} = \dfrac{2 \times \text{area of } \Delta A'B'C'}{2 \times \text{area of } \Delta ABC} = k$, and $a' = kd$. Similarly $b' = ke$, $c' = kf$.

Thus the sides of $\Delta A'B'C'$ must be proportional in length to the altitudes of the given triangle. The desired construction may be accomplished by constructing the altitudes of the given triangle, then a triangle whose sides are equal in length to those altitudes. If this triangle is A'B"C", (with B"C" the side of length d) construct the altitude A'M and the point N on A'M such that A'N = BC. Construct B'C' parallel to B"C" to complete the construction of the desired triangle A'B'C'.



Q.558. Show that 3 is the only positive integer value of n which is such that 2n-1 is a factor of $(3n^2 - 3n + 1)(3n^2 - 3n + 2)$.

Answer There is actually another obvious value of n, namely n = 1, for which 2n - 1 is equal to 1 and hence is trivially a factor of any integer.

Note that $3(2n - 1)^2 = 12n^2 - 12n + 3$ which is clearly relatively prime to $12n^2 - 12n + 4 = 4 \times (3n^2 - 3n + 1)$, and any common factor which it has with $12n^2 - 12n + 8$ (i.e. $4 \times (3n^2 - 3n + 1)$) must be a divisor of the difference, viz. 5. Thus 2n - 1 cannot be a factor of $(3n^2 - 3n + 1)(3n^2 - 3n + 2)$ unless 2n - 1 = 5, or 1 viz. n = 3, or 1. Correct Solution Phuong - V - Nguyen (Fairfield H.S.).

Q.559. If the numbers 1,2,3,.....,N are placed in N small hexagons fitted together in a hexagonal pattern in such a way that sums along rows in any of the three directions through each small hexagon always come to the same total, the resulting arrangement is called a "magic hexagon".

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        15   13   10
     14    8    4    12
   9    6    5    2    16
     11    1    7    19
        18   17    3
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The diagram shows a magic hexagon of order 3 (there are three small hexagons along a side of the hexagonal pattern). Totals along rows in any direction are all equal to 38.

(a) Show that there is no magic hexagon of order 2.

(b) Find N (the number of small hexagons) in the hexagonal pattern of order n, and find also the total along rows in a magic hexagon of order n.

(c) Show that if $n > 3$ there is no magic hexagon or order n.

(The result of Q.558 may be helpful).

Answer

(a)

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   x   y
 z   t   u
   v   w
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Since x,y,z must be three different numbers from the list 1,2,3,4,5,6,7, $y \neq z$. However, in a magic hexagon $x + y = x + z$, yielding $y = z$. Contradiction.

(b) One way to calculate N is from the equation $N = 1 + 6 \times 1 + 6 \times 2 + 6 \times 3 + \dots\dots + 6 \times (n - 1)$. The first term counts the central hexagon, the next term the 6 hexagons which touch it, then the twelve hexagons in the next layer out, and so on. Alternatively, summing by horizontal rows one would obtain $N = n + (n+1) + (n+2) + \dots\dots + (2n - 2) + (2n - 1) + (2n - 2) + \dots\dots + (n+1) + n$.

Whichever of the two expressions is taken, the formula for summing an arithmetic progression can be used to simplify it to

$$N = 3n^2 - 3n + 1 .$$

In a magic hexagon of order n the total of the numbers in all squares $1 + 2 + 3 + \dots\dots + N = \frac{1}{2}N(N + 1) = \frac{1}{2}(3n^2 - 3n + 1)(3n^2 - 3n + 2)$. There are $2n - 1$ horizontal rows in the figure, so the sum of the entries in each row must be

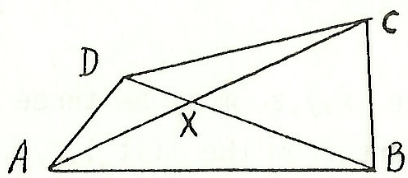
$$\frac{3n^2 - 3n + 1)(3n^2 - 3n + 2)}{2(2n - 1)}$$

(c) By Q.558, if $n > 3$ $\frac{(3n^2 - 3n + 1)(3n^2 - 3n + 2)}{2(2n - 1)}$ cannot be a whole number, so it is impossible that any magic hexagon of order n exists. Correct Solution J. Hardy (Darmalan College).

Q.560. For an integer $n > 1$, let $f(n)$ be the product of all the positive divisors of n other than n itself. Find a simple description of all numbers n such that $f(n) = n$.

Answer Suppose d is a factor of n with $1 < d < \sqrt{n}$. Then $d' = n/d$ is another factor of n, and dd' is already equal to n. Hence if any other divisor of n greater than 1 exists we would have $f(n) > n$. Thus for $f(n) = n$ there is only one divisor of n in the range $1 < d < \sqrt{n}$, (and d must therefore be a prime number), and one in the range $\sqrt{n} < d' < n$. There are two possibilities for d'; either it has no factors other than itself and unity i.e. it is a prime, or its only smaller factor is d, i.e. $d' = d^2$. Thus $n = f(n)$ if n is the product of two different primes, or is the cube of a prime number.

Q.561.



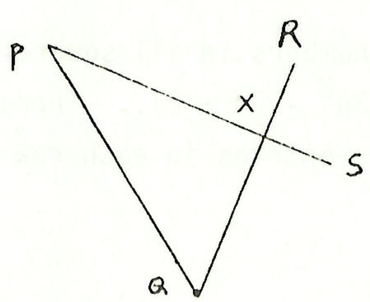
The figure shows a convex quadrilateral with both diagonals. There are eight triangles whose sides are sides of the quadrilateral, or diagonals, or segments of diagonals (viz ABC, BCD, CDA, ABD, ABX, BCX, CDX and ADX).

If all five diagonals are drawn in a convex pentagon there are 35 such triangles.

If no three diagonals of a convex n-gon are concurrent, find a formula for the number of triangles whose sides are all either sides of the n-gon, or diagonals of the n-gon, or segments of the diagonals.

Answer There are ${}^nC_3$ ways of choosing three vertices of the given n-gon, and each such choice determines one of the triangles to be counted.

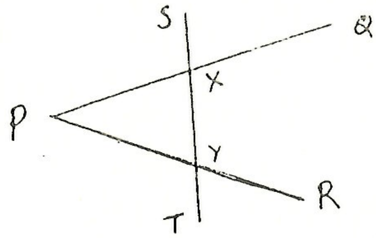
Others of the triangles have two of the vertices coincident with 2 of the vertices of the n-gon (P and Q, say), the third vertex, X, being the intersection of diagonals PS and QR.



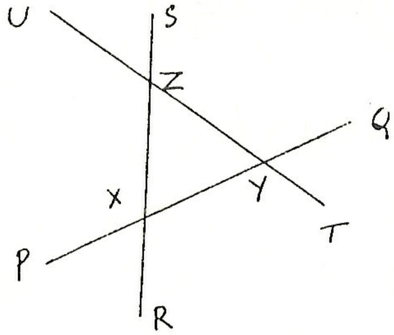
There are ${}^nC_4$ ways of choosing, four vertices of the given n-gon, and then 4 ways of choosing one neighbouring pair of these to be the side PQ.

Thus there are $4 \times {}^nC_4$ triangles of this type to be counted.

Another class consists of triangles with one vertex (P say) a vertex of the n-gon and the other two vertices X and Y intersections of a diagonal S X Y T with diagonals PQ and PR. There are $^nC_5$ ways of selecting 5 vertices of the given n-gon, and then 5 ways to choose one of them to be labelled P. After that the triangle P X Y is uniquely determined. Hence there are $5 \times {}^nC_5$ triangles of this type.



Finally there are triangles all of whose vertices are inside the given n-gon, having sides lying in three diagonals PQ, RS, and TU. There are $^nC_6$ ways of choosing 6 vertices of the given n-gon and for each such choice only one triangle X Y Z can result. Thus there are $^nC_6$ triangles of this type.



The total number of triangles for any convex n-gon is thus given by

$$^nC_3 + 4\,{}^nC_4 + 5\,{}^nC_5 + {}^nC_6$$

$$= \frac{n(n-1)(n-2)}{3!} + 4\,\frac{n(n-1)(n-2)(n-3)}{4!} + 5\,\frac{n(n-1)(n-2)(n-3)(n-4)}{5!} +$$

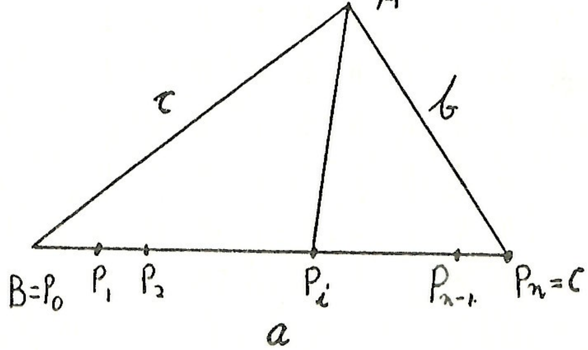
$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!}$$

$$= \frac{n(n-1)(n-2)}{720}\,[n^3 + 18n^2 - 43n - 60]$$

Q.562. (Submitted by Howard See.)

Divide the side BC in ΔABC into n equal intervals $P_{i-1}P_i$, $i = 1,2,\ldots,n$ where P_0 coincides with B and P_n with C. Find (in terms of the side lengths a, b, c).

$$\lim_{n\to\infty} \frac{1}{n}\left(AP_1^{\,2} + AP_2^{\,2} + \ldots\ldots + AP_n^{\,2}\right).$$



Answer. In ΔABP_i, $BP_i = i\frac{a}{n}$. For $i = 1, 2, \ldots n$.

$$AP_i^2 = AB^2 + BP_i^2 - 2AB\ BP_i \cos B$$

$$= c^2 + \frac{i^2a^2}{n^2} - \frac{2ac \cos B\ i}{n}$$

$$\therefore \frac{1}{n}\sum_{i=1}^{n} AP_i^2 = \frac{1}{n}\left[nc^2 + \frac{a^2}{n^2}\sum_{i=1}^{n} i^2 - \frac{2ac\cos B}{n}\sum_{i=1}^{n} i\right]$$

$$= c^2 + \frac{a^2}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{2ac\cos B}{n^2}\frac{n(n+1)}{2}$$

$$= c^2 + \frac{a^2}{6}(1+\frac{1}{n})(2+\frac{1}{n}) - ac\cos B(1+\frac{1}{n})$$

$$\therefore \lim_{n\to\infty}\frac{1}{n}\sum AP_i^2 = c^2 + \frac{a^2}{3} - ac\cos B$$

$$= c^2 + \frac{a^2}{3} - ac\frac{(a^2+c^2-b^2)}{2ac}$$

$$= \frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{6}.$$

J. Hardy contributed a correct solution using calculus methods.

Solvers of earlier problems:

Good solutions of problems Q539, Q540, Q541, Q545, Q346, and Q549 by Andrew & Richard Stone (Canberra Grammar School) were received too late for acknowledgement in the last issue.

I owe an apology to each of the following two problem solvers whose work I could well have used a couple of issues back if it had not got lost in my filing system.

Tim Bedding (Beacon Hill High School) Q527, Q528, Q531.

Marfus Brazil (Canberra Grammar School) Q527, Q528, Q531, Q532, Q538.