# H.S.C. CORNER BY TREVOR

In the 1983 HSC 4 unit (Unique) paper there were some interesting problems on functions and their graphs. The first problem is pretty easy:

Problem 84.1. Draw a careful sketch of the curve $y = x^2/(x^2 - 1)$, indicating clearly any vertical or horizontal asymptotes, turning points, or inflexions.

Solution. There are standard procedures for sketching a curve like this, as follows:

a) Intercepts: Inspection shows that the curve intersects the $x,y$ axes only at $(0,0)$.

b) Symmetry: Replacing $x$ by $(-x)$ gives the same curve. Thus the curve is symmetric about the $y$ axis.

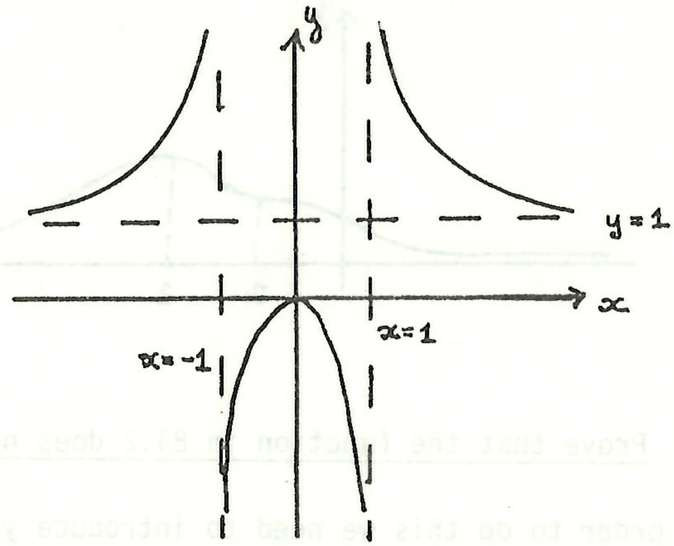
c) Turning points: $dy/dx = -2x/(x^2-1)^2$, and $d^2y/dx^2 = -2(3x^2+1)/(x^2-1)^3$. Hence the only turning point is at $(0,0)$, and this is a maximum. Also the second derivative is never zero, so there are no inflexion points.

d) Asymptotes: There are asymptotes at $x = \pm 1$. Also, $\lim_{x\to\infty} y = 1$, and hence $y = 1$ is an asymptote.

e) Parity: Note that $y = x^2/(x-1)(x+1)$, and hence construct the following table

| $x < -1$ | $-1 < x < 1$ | $x > 1$ |
|---|---|---|
| $y > 0$ | $y < 0$ | $y > 0$ |

The sketch incorporating all these features is:




Graph of
$y = x^2/(x^2-1)$.

Problem 84.2. A function $f(x)$ is known to approach 0 as $x$ approaches $\infty$ and $-\infty$. Its derivative is given by

$$f'(x) = e^{-x^2}(x-1)^2(2-x).$$

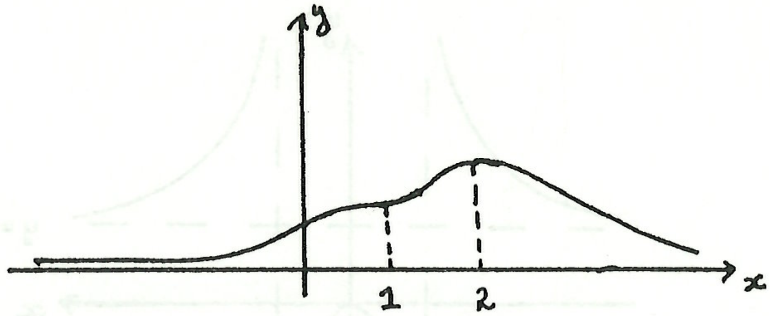
From this information, describe the behaviour of $f(x)$ as $x$ increases from $-\infty$ to $+\infty$. Include in your description an indication of those $x$ where $f(x)$ is respectively increasing or decreasing and any points where $f$ has a maximum value. Also explain why $f(x)$ must be positive for all real $x$. Draw a sketch of a function $f(x)$ satisfying the given conditions.

Solution: The turning points are when $f'(x) = 0$, i.e. when $x = 1$, and $x = 2$. Also note that $e^{-x^2} > 0$, $(x-1)^2 \geqslant 0$, and thus

$$f'(x) \geqslant 0 \text{ for } x < 2, \text{ and } f'(x) < 0 \text{ when } x > 2.$$

It follows that there is a horizontal inflexion point at $x = 1$, and a maximum at $x = 2$. The line $y = 0$ is also an asymptote, where $y = f(x)$.

Moreover, since there is only one turning point, at $x = 2$, and $f'(x) \geqslant 0$ for $x < 2$, and $f'(x) < 0$ for $x > 2$, it follows that $f(x) > 0$ for all finite $x$. The sketch must look like this




BUT

Problem 84.3. Prove that the function in 84.2 does not exist!

Solution: In order to do this we need to introduce you to a new notation:

Definition: $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx,$

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx,$$

$$\int_{-\infty}^\infty f(x)dx = \lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b f(x)dx,$$

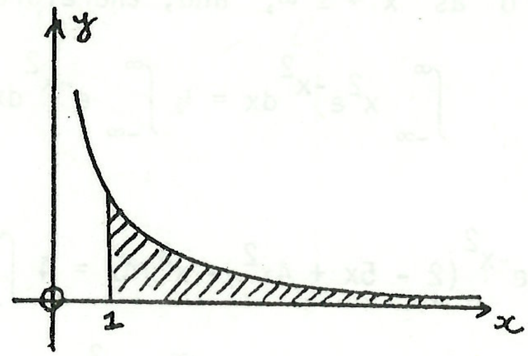
provided these limits exist. For example

$$\int_1^A \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^A = 1 - \frac{1}{A}.$$

But $\frac{1}{A} \to 0$ as $A \to \infty$, . . ., and hence,

$$\int_1^\infty x^{-2}dx = \lim_{A\to\infty} (1 - A^{-1}) = 1.$$

Consider the sketch of $y = x^{-2}$ for $x > 1$.




The area under the curve is given by $\int_1^{\infty} x^{-2}dx$.

Now assume that the function in 84.2 $\to 0$ as $x \to -\infty$. Given that

$$\frac{dy}{dx} = e^{-x^2}(x - 1)^2(2 - x),$$

it follows that, since $y \to 0$ as $x \to -\infty$,

$$y = \int_{-\infty}^{x} e^{-x^2}(x - 1)^2(2 - x)dx.$$

Hence, as $x \to \infty$,

$$y \to \int_{-\infty}^{\infty} e^{-x^2}(x - 1)^2(2 - x)dx = \int_{-\infty}^{\infty} e^{-x^2}(2 - 5x + 4x^2 - x^3)dx.$$

Now notice that when $n$ is an odd integer, $x^n e^{-x^2}$ is an <u>odd</u> function of $x$.

Hence, for odd $n$, $\int_{-\infty}^{\infty} x^n e^{-x^2}dx = 0$. So

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{\infty} x^3 e^{-x^3}dx = 0.$$

Also, using integration by parts,

$$\int x^2 e^{-x^2}dx = \tfrac{1}{2}\int xe^{-x^2}d(x^2) = -\tfrac{1}{2}\int xd(e^{-x^2})$$

$$= -\tfrac{1}{2}xe^{-x^2} + \tfrac{1}{2}\int e^{-x^2}dx$$

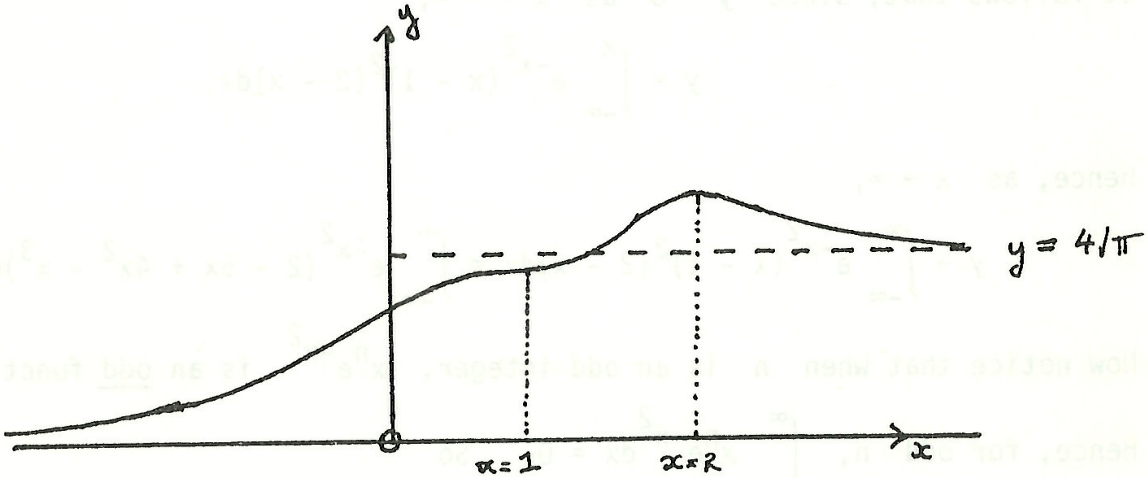
Now note that $xe^{-x^2} \to 0$ as $x \to \pm\infty$, and, therefore,

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx.$$

Thus,

$$\int_{-\infty}^{\infty} e^{-x^2}(2 - 5x + 4x^2 - x^3) dx = 4 \int_{-\infty}^{\infty} e^{-x^2} dx.$$

Finally, it is a well-known result that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Hence, if we assume that, in Problem 84.2, $y \to 0$ as $x \to -\infty$, it necessarily follows that $y \to 4\sqrt{\pi}$ as $x \to \infty$, and the line $y = 4\sqrt{\pi}$ is an asymptote as $x \to \infty$. The correct sketch is:




Problem 84.4. A plane curve is defined implicitly by the equation

$$x^2 + 2xy + y^5 = 4.$$

This curve has a horizontal tangent at the point $P(X, Y)$. Show that $X$ is the unique real root of the equation $X^5 + X^2 + 4 = 0$, and that $-2 < X < -1$.

Solution: We start this problem by differentiating implicitly to give

$$2x + 2x\frac{dy}{dx} + 2y + 5y^4\frac{dy}{dx} = 0. \tag{2}$$

At a turning point $(X,Y)$, $\frac{dy}{dx} = 0$, so that it follows from (2) that $X + Y = 0$, i.e. $Y = -X$. Substitute this result in (1), to obtain

$$X^2 - 2X^2 - X^5 = 4,$$

i.e. $X$ must satisfy the quintic equation

$$g(x) = x^5 + x^2 + 4 = 0. \qquad (3)$$

Note that $g \to \infty$ as $x \to \infty$, and $g \to -\infty$ as $x \to -\infty$. Since all polynomials are continuous, it follows that the quintic equation (3) has at least one real root.

We now prove that there is only one real root as follows:

$$g'(x) = 5x^4 + 2x = x(5x^3 + 2),$$

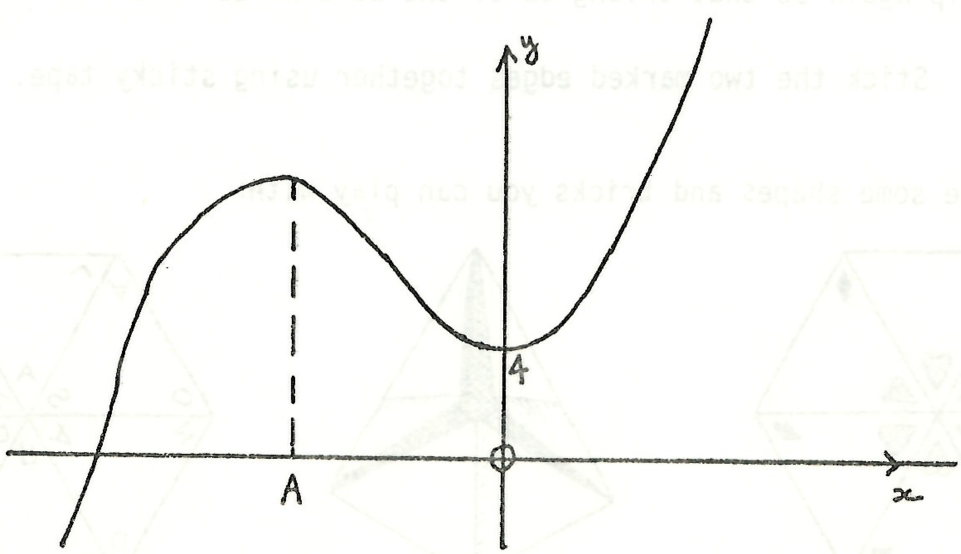
$$g''(x) = 20x^3 + 2.$$

The zero of $g'(x)$ are at $x = 0$, $x = A = -(\frac{2}{5})^{1/3}$, and then $g''(0) = 2 > 0$, and $g''(A) = 20A^3 + 2 = -6 < 0$. Thus $g(x)$ has a minimum at $x = 0$, and a maximum at $x = -(\frac{2}{5})^{1/3}$. Also $g(0) = 4$, and $g(A) = A^5 + A^2 + 4$,

$$= -\frac{2}{5}A^2 + A^2 + 4,$$

$$= \frac{3}{5}(\frac{2}{5})^{2/3} + 4 > 4.$$

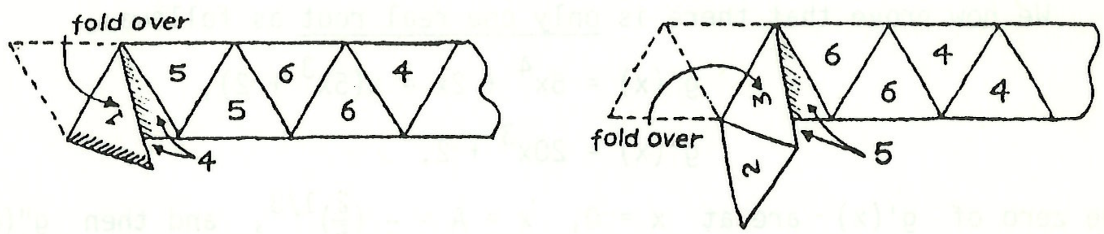
Thus both the turning points are above the $x$ axis and the function must look some thing like this:



Obviously $g(x) = 0$ has only one root, at $x = X$. Also $g(-1) = -1 + 1 + 4 = 4$, $g(-2) = -32 + 4 + 4 = -24 < 0$. Hence $g(-2) < 0$ and $g(-1) > 0$. Since $g(x)$ is continuous it follows that the root of $g(x) = 0$ must lie between $x = -2$ and $x = -1$. Thus $-2 < X < -1$.

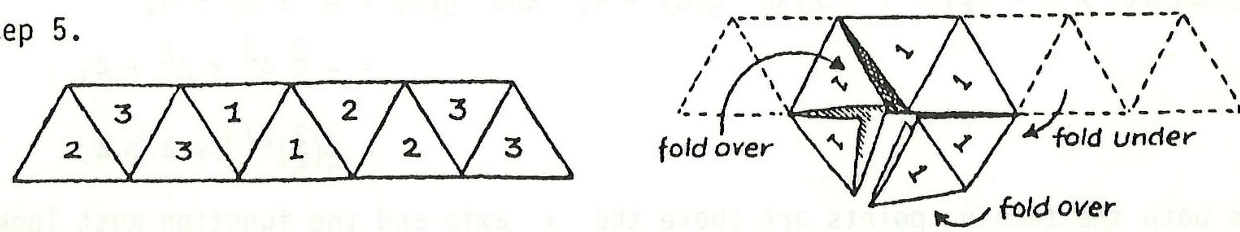
*(continued from page 14.)*

Step 4.




Fold the strip matching triangles of same number.

Step 5.




Fold the strip again so that triangles of the same number are all on top.

Step 6. Stick the two marked edges together using sticky tape.

Here are some shapes and tricks you can play with: