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A COMBINATORIAL APPROACH TO GOLDBACH'S CONJECTURE

BY

J.H. CLARKE AND A.G. SHANNON *

In 1742 Goldbach suggested in a letter to Euler that every even integer greater than 4 is the sum of two odd primes. Although many attempts have been made to establish this conjecture, a proof is still awaited. The "circle method" of Hardy and Littlewood [1] showed that the method was tractable. By using a cominatorial approach which can be further extended by the interested reader, it is demonstrated here that it is highly unlikely that Goldbach's conjecture is false.

Our approach is best introduced by an example. Consider 18 and tabulate all the ways of adding two odd integers greater than 1 to give 18:

3	15
5	13
7	11
9	9

Assume that there are h_1 primes in the first column and h_2 primes in the second (3 and 2 in our example). If a prime in the first column is adjacent to a prime in the second, then Goldbach's conjecture works for this particular number (7 and 11 in our case).

Given two adjacent columns of r entries in how many ways it is possible to choose h_1 in the first column and h_2 in the second so that no two of the chosen ones are adjacent? Having chosen the h_1 the other h_2 may be chosen in $\begin{pmatrix} r - h_2 \\ h_2 \end{pmatrix}$ ways, and so the total number of ways of choosing the h_1 and h_2

* Dr. Clark is an Honorary Associate and Dr. Shannon a Senior Lecturer in Mathematics at the New South Wales Institute of Technology.

without adjacent choices is

$$\binom{r}{h_1}\binom{r - h_1}{h_2}.$$

On the other hand the number of ways of freely choosing h_1 in the first column and h_2 in the second is

$$\binom{r}{h_1}\binom{r}{h_2}.$$

So if, in our set-up, there are r numbers in each column, the probability of violating Goldbach's conjecture is

$$\binom{r}{h_1}\binom{r - h_1}{h_2}\Big/\binom{r}{h_1}\binom{r}{h_2}$$

i.e.
$$\binom{r - h_1}{h_2}\Big/\binom{r}{h_2}.$$

Goldbach's conjecture has been verified by a computer study to be correct for all even integers up to 10^8. And, for large numbers, $h_1 \approx h_2$, as indicated by Hardy and Wright [2]. For example, the number of primes less than 5000 is 667 while the number of primes between 5000 and 10000 is 571. So, for large r, the probability of violating Goldbach's conjecture is given by approximately

$$\binom{r - h}{h}\Big/\binom{r}{h} = \frac{(r - h)(r - h - 1)\ \dots\ (r - 2h + 1)}{r(r - 1)\ \dots\ (r - h + 1)}$$

$$= \left(1 - \frac{h}{r}\right)\left(1 - \frac{h}{r - 1}\right)\ \dots\ \left(1 - \frac{h}{r - h + 1}\right)$$

$$< \left(1 - \frac{h}{r}\right)^h$$

For example the number of primes less than 10^6 is 78498 and so considering 10^6 in this way gives $r \approx 250000$ and $h \approx 39250$. So the probability in this case is less than

$$\left(1 - \frac{39250}{250000}\right)^{39250} \approx 10^{-2911}$$

This is an exceedingly low probability, so low in fact that only a pure mathematician would not consider the event to be impossible!

Further investigation suggest themselves as exercises. For instance, by utilising the results of Rosser and Schoenfeld [4], that if $\pi(x)$ is the number of primes less than x, then

$$x/\log x < \pi(x) \quad \text{for} \quad x > 17$$

and

$$3x/(5 \log x) < \pi(2x) - \pi(x) \quad \text{for} \quad x \geqslant 21,$$

one can test the claim that the number of primes less than x is approximately the same as the number of primes between x and $2x$ as x increases. Finer estimates of (computer- or calculator-generated) numerical data would also permit tabulations of the probability.

Further relevant exercises can be obtained by extending some of the ideas in Mirsky's historical survey [3] of the problem.

References.

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3. L. Mirsky, "Additive prime number theory", Math. Gaz. 42 (1958).
4. J. Barkley Rosser and Lowell Schoenfeld, "Approximate formulas for some functions of prime numbers", Illionois J. Math. 6 (1962).