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MINIMUM ROADWAY PROBLEMS

BY

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1. Introduction

One of the mathematical results that one encounters early in life is that concerning the shortest path joining two points. It is, of course, the straight line. Any other path will have a greater length. If we attempt to generalize this result, in order to determine the minimum path connecting three, or four, or more points, the problem becomes rapidly more complicated. These problems, however, are important in such applications as the construction of roadways connecting a number of towns, and pipelines or cables joining a number of centres. In all these cases the cost of construction is proportional to the length, so minimizing the length also minimizes the cost.

Let us restrict the discussion to the construction of roadways. Can we derive any properties of minimum roadway configurations, linking a number of towns, before attempting the more difficult task of finding complete solutions? If we consider a number of towns connected by some roadway system (see Figure 1(a)) it is clear that the minimum roadway configuration cannot have any curved roads, as any curved section of road can be replaced by a shorter straight-line length of road. When all the curved roads are replaced by straight-line roads, we obtain the configuration shown in Figure 1(b).

2. Three-town problems

In order to gain further insight into the general problem, let us examine the three-town problems. The simplest problem would seem to occur when all the towns, A, B and C, say, are arranged at the vertices of an equilateral triangle (see Figure 2) with sides of length d. The minimum roadway system might be

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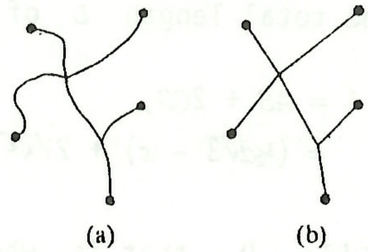

Figure 1. Roadway configurations; the straight-line road system has the shorter length.

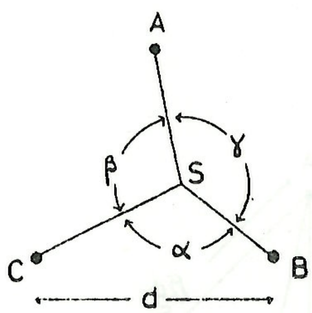

Figure 2. Three towns arranged at the vertices of an equilateral triangle.

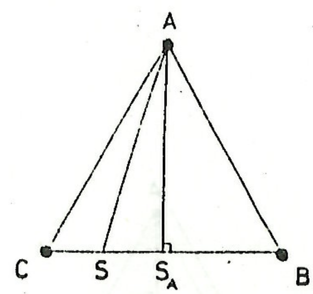

Figure 3. A roadway joining A, B and C with S along BC.

along two sides - with $L = 2d$ - or along three straight-line roads meeting at a point S with angles between the roads of α, β, γ, as indicated in Figure 2 - or one side and one road joining the opposite vertex to that side, as in Figure 3. The minimum length in this case is clearly when S is at S_A, the midpoint of CB. This gives $L = (1 + \frac{1}{2}\sqrt{3})d = (1.866\ldots)d$, which is clearly shorter than the wo-side solution with $L = 2d$.

We might now look at a three-road system with S on AS_A, as indicated in Figure 4. If $SS_A = x$ then the total length L of the roadway is given by

$$L = AS + 2CS, \tag{1}$$

$$= (\tfrac{1}{2}d\sqrt{3} - x) + 2\sqrt{x^2 + \tfrac{1}{4}d^2}\ . \tag{2}$$

This will be minimized when $dL/dx = 0$, that is, when

$$0 = -1 + 2x(x^2 + \tfrac{1}{4}d^2)^{-\frac{1}{2}}, \tag{3}$$

or

$$x = {}^1/_6\sqrt{3}d. \tag{4}$$

Consequently $\alpha = 120^0$, and, by symmetry, $\gamma = \beta = 120^0$. The value of this minimum is, from (2), $\sqrt{3}d$.

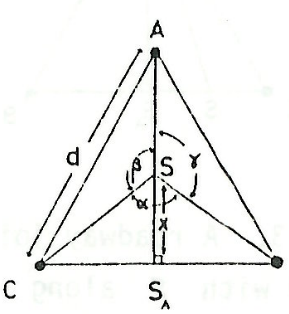

Figure 4. A roadway configuration with S along AS_A.

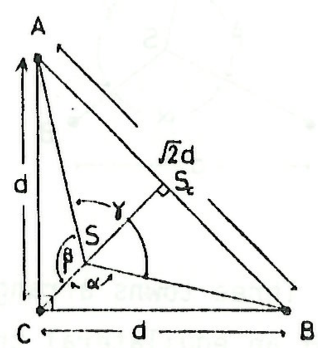

Figure 5. Three towns arranged at the vertices of an isosceles right-angled triangle.

This argument does not prove that this is *the* minimum roadway, but indicates that this may well be the case. It is certainly shorter than any system with S on the boundary or at any other point on the bisector of an angle of the triangle. Further analysis will be needed to prove that it is *the* true minimum.

Before attempting a general proof for the three-town problems, let us examine a further example. Consider the towns at the corners of an isosceles right-angled triangle with sides of length d and $\sqrt{2}d$, as in Figure 5. Similar arguments to those presented for the equilateral triangle would indicate that the position of S might be along the bisector CS_c of $\hat{C}$ where S_c is the mid-point of AB. We can obtain a general expression for the total length $L(x)$ of the roadway for S along CS_c as a function of $x = SS_c$, and minimize L with respect to x. Again we obtain $\gamma = \alpha = \beta = 120^0$, the total length of roadway being $\frac{1}{2}(\sqrt{6} + \sqrt{2})$.

At this stage we might conjecture that, for any arrangement of three towns, the minimum path has this 120^0 property. We shall now show that this is indeed the case, using a geometrical proof that was brought to my attention by Mr. R.D. Nelson of Ampleforth College, York (see reference 2).

Consider any triangle ABC with a 'roadway' system of length L formed by three lines meeting at S, as in Figure 6. Now rotate the shaded triangle anticlockwise about C through 60^0, so that A is now at A' and S' is now at S'. Then $S'C = SC$ and $S'CS = 60^0$, so triangle CSS' is an equilateral triangle. Also,

$$L = AS + BS + CS = A'S' + SS' + SB \qquad (5)$$

as, by definition, $A'S' = AS$ and $S'S = SC$, $CS'S$ being an equilateral triangle. Now A' and B are fixed points. As S is varied,

$$L = A'S' + S'S' + SB$$

will be minimized when $A'B$ is straight line (see Figure 7). In Figure 7, $\alpha = \widehat{CSB}$ and $S'SB$ is a straight line, so

$$\alpha = 180^0 - 60^0 = 120^0 \qquad (6)$$

Also, using the notation in Figure 7,

$$\begin{aligned} \beta &= \beta', && \text{by definition,} \\ &= 180^0 - 60^0, && \text{as } A'B \text{ is a straight line,} \\ &= 120^0. && (7) \end{aligned}$$

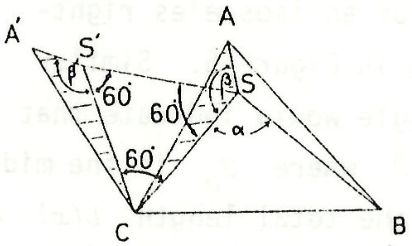

Figure 6. *CAS* rotated through 60^0 about *C*.

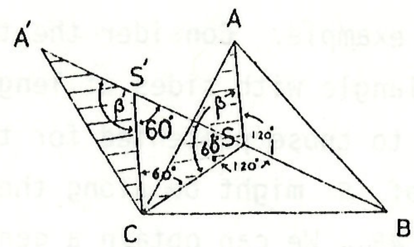

Figure 7. The minimum-path position for *S*.

Hence, from (6) and (7),

$$\gamma = 360^0 - \alpha - \beta - 120^0. \quad (8)$$

Thus the 120^0 angles are a general property of the minimum path connecting *A*, *B* and *C*.

These results can be demonstrated experimentally by drilling three small holes in a horizontal wooden table. They represent the three points *A*, *B* and *C*. Three equal weights mg are now hung below the holes from strings that are tied together on the table, the height of the weights above the ground being h_1, h_2 and h_3. The three weights will come to equilibrium when their total potential energy is minimized. This will occur when $mg(h_1 + h_2 + h_3)$ is minimized, that is, when $(h_1 + h_2 + h_3)$ is minimized. This will clearly occur when the sum of the three horizontal lengths of string on the table is minimized. So the final equilibrium configuration of the horizontal strings is one in which their total length is minimized. Also, as the tension in each string is mg, the equilibrium configuration occurs when pairs of adjacent strings intersect at 120^0.

These proofs, however, are only valid provided that the angles of the triangle are less than or equal to 120^0, when *S* lies inside or on the triangle.

For triangles with an angle equal to 120°, the minimum path length is the sum of the two shortest sides of the triangle, i.e. those adjacent to the largest angle. This latter result can also be proved to hold for triangles with an angle greater than 120°. Summarizing, a triangle with no angle greater than 120° has a minimum path formed by three lines meeting at a point S inside the triangle at 120°. All other triangles have a minimum path formed by the two sides adjacent to the largest angle of the triangle.

3. Many-town problems

We can show, using the result for the minimum roadway joining three towns, that the general minimum roadway connecting a number of towns can only have roadway intersections with three roads, each road making an angle of 120° with the adjacent roads.

If the general minimum roadway consisted of a roadway intersection with more than three roads, as in Figure 8(a), one could, using the results for three points, find a roadway configuration with a smaller length. For example, in Figure 8(b), using points P and Q in triangle POQ, with $\widehat{POQ} < 120^{\circ}$, we can form a roadway configuration with a small length by replacing the full lines PO and QO by the broken lines, PS, SO and SQ, meeting at 120°. Repeating this procedure with all the roads meeting at O, we finally obtain only three

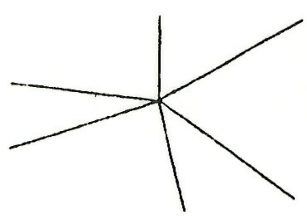

Figure 8(a). Intersection point with more than three roads.

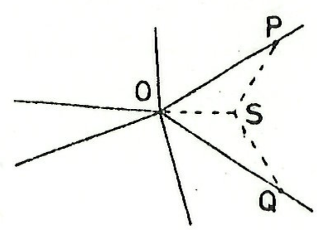

Figure 8(b). A roadway with a shorter length than that in Figure 8(a).

roads meeting with angles of 120°.

These minimum-path, or roadway, problems were investigated during the nineteenth century by the Swiss mathematician Jacob Steiner (Figure 9). The points at the three-way intersection are often known as *Steiner points*.

Let us examine the four-town problem with the four towns arranged in a square array with sides of length d (Figure 10(a)). On the grounds of symmetry, we might be tempted to guess that the minimum roadway has an X-configuration, with length $2\sqrt{2}d$. However, we know that the roadway intersection must consist of three roads meeting at 120°. Consequently, the only possible solutions are the two shown in Figure 10(b) by the full and broken lines with angles 120°. These give roadways of length $(1 + \sqrt{3})d = (2.73..)d$, which is shorter than the X-configuration of length $2\sqrt{2}d = (2.82...)d$. It is of interest to investigate how the minimum roadway configurations alter when the towns are arranged in a rectangular array with $AD = d$ and $AB = (1 + \omega)d$. If ω is increased from 0 the two configurations in Figure 11 will have different lengths, one being a local minimum and the other an absolute minimum. The length of the full road is

Figure 9. Jacob Steiner (1796 - 1863)

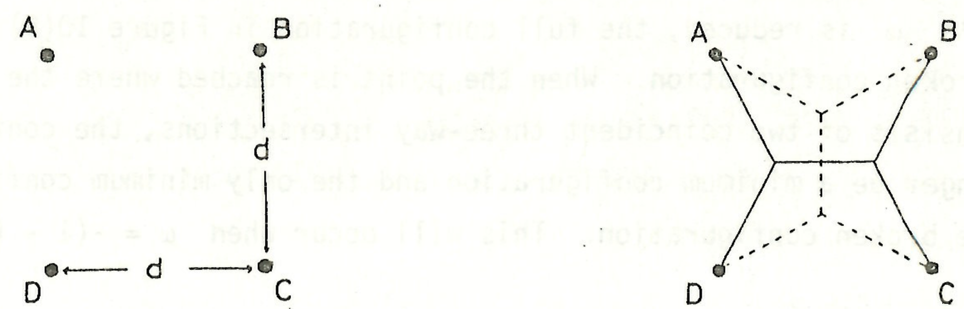

Figure 10. (a) The square array of four towns. (b) The minimum roadway configurations.

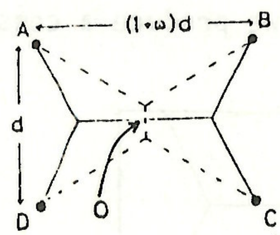

Figure 11. The two minimum roadways for a rectangular array of towns.

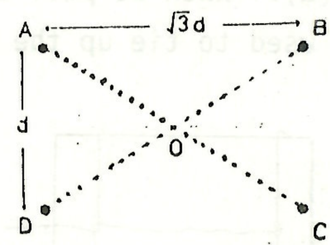

Figure 12. The configuration with the critical value of $\omega = (\sqrt{3} - 1)d$.

$(1 + \sqrt{3} + \omega)d$, and that of the broken road $(1 + \sqrt{3}(1 + \omega))d$. Consequently the broken road is the local minimum. As ω is increased, the two three-way broken intersections will eventually meet, leading to a four-way intersection at O. in Figure 11. When this occurs $\omega = \sqrt{3} - 1$, and the configuration will no longer be a minimum configuration as we can form a roadway with a shorter length by replacing AO and DO by three roads meeting at 120^0 as indicated by the dotted lines in Figure 12. This configuration is the same as the absolute minimum. So, for $\omega \geq (\sqrt{3} - 1)$ there is only one minimum roadway. A similar situation arises when $\omega < 0$, the rectangular array of towns being such that $AB < AD$. As ω is reduced, the full configuration in Figure 10(b) is longer than the broken configuration. When the point is reached where the full configuration consists of two coincident three-way intersections, the configuration will no longer be a minimum configuration and the only minimum configuration will be the broken configuration. This will occur when $\omega = -(1 - (\sqrt{3}/3))$.

4. Analogue methods and soap films

It is perhaps unexpected to find that the square array of towns has a minimum roadway configuration that does not have the full symmetry of the square. However, it should not be thought so unusual, as we regularly encounter the solution to this problem in another context.

We solve this problem every time we tie up a parcel with string. Usually we start by wrapping the string around the parcel with the configuration shown in Figure 13(a). When we pull on the string, in order to minimize the length of string being used to tie up the parcel securely, we produce a constant tension

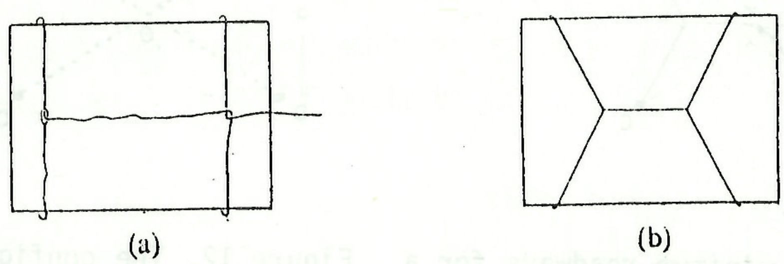

Figure 13. Tying up a parcel: (a) initial configuration, string loose (b) string taut.

in the string. This leads to the configuration in Figure 13(b). The three-way string-intersection has three equal forces, produced by the constant tension in the string, in equilibrium, so three strings must meet at 120^0. Thus minimizing the length of string leads to all the features associated with minimum roadway problems.

Another analogous system, which has the advantage that it can easily be applied to any number of points, is that based on the properties of soap films (see references 1 and 3). A soap film has the property that its energy is proportional to its area. For example, a soap film contained by a circular ring will not bulge out at equilibrium but will form the minimum-area surface, the disc contained by the circular ring.

In order to make use of this minimum-area property to solve minimum-path problems, we must convert the minimum-area property into a minimum-path property. To do this, let us first focus attention on the simplest problem: the minimum path connecting two points. Consider a soap film contained between two parallel clear perspex plates, with two pins perpendicular to the plates, separating the plates, and at some distance from each other. Then, by symmetry, the film will be perpendicular to the plates and bounded by the two plates, beginning on one pin and ending on the other (Figure 14). Consequently the film will be in the form of a tape, with constant width equal to the distance between the plates. The area of the soap film is proportional to its length. When it comes to equilibrium it will have minimum area and also minimum length. Thus the tape will end up, in equilibrium, with the straight-line configuration in Figure 14. The analogous solution can, by the same reasoning, be extended to any arrangements of pins and points. The soap film solution to the four-town problem is shown in Figure 15.

We have seen that the four-town problem can have two minimum configurations. In order to determine the minimum roadway with the smallest length we need to calculate the length of each path and determine the one with the smallest length using the 120^0 property. In problems with many towns there may be many minimum configurations. This analogue method is based on producing soap films between the plates by dipping the plates into a bath of soap solution at different angles or perturbing an equilibrium soap film by blowing it into another equilibrium configuration. There is no simple method of determining analytically all the minimum configurations (see reference 4).

An interesting example of a problem with three minimum configurations occurs when solving the minimum path joining six towns arranged at the vertices of a regular hexagon. The three configurations, which can easily be obtained using soap films, are shown in Figure 16.

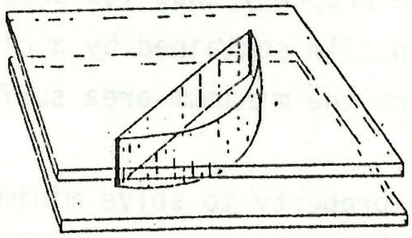

Figure 14. Soap films bounded by parallel plates and two pins. The curved surface is a non-equilibrium soap film and the straight surface is the minimum-area, minimum-path, surface.

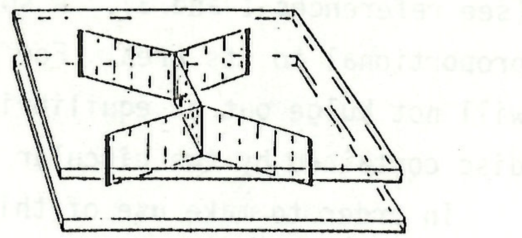

Figure 15. The minimum-path soap film joining four pins arranged in a square array.

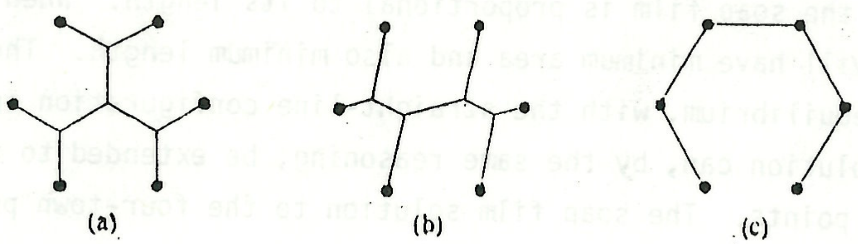

Figure 16. Minimum configuration for six towns arranged in a regular hexagon.

If the sides of the hexagon have unit length, the lengths of the minima can be calculated, using the 120^0 property, to be $3\sqrt{3}$, $2\sqrt{7}$ and 5 respectively. It is interesting to note that the configurations have, 3, 2 and 1-fold symmetry about the axis of symmetry perpendicular to the plane of the hexagon. The configuration of smallest length in this case is 16(c). However, in other problems it may well be one of the internal roadways.

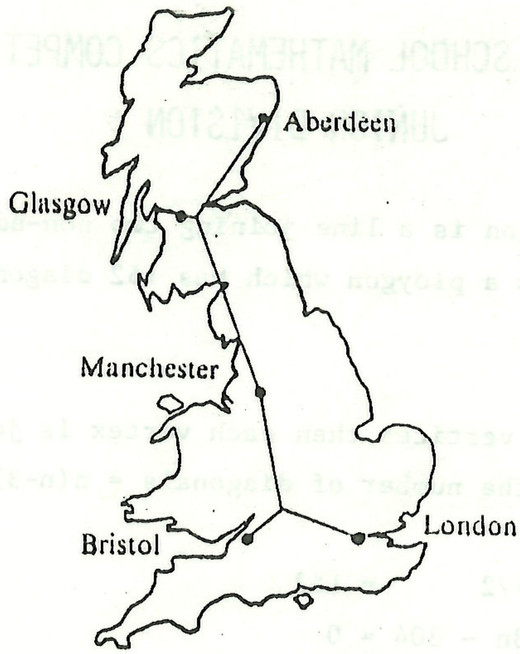

Figure 17. The minimum roadway system linking London, Bristol, Manchester, Glasgow and Aberdeen.

Let us now apply this analogue method to the practical problem of linking London, Bristol, Manchester, Glasgow and Aberdeen by the shortest length of roadway. It is necessary to draw a map of Britain on one of the parallel perspex plates and insert pins perpendicular to the plates at these towns. After dipping the plates into soap solution we obtain the roadway configuration shown in Figure 17, with one Steiner point to the east of Glasgow and one south of Birmingham. In this application, there is only one minimum roadway configuration.

References

1. R. Courant and H. Robbins, *What is Mathematics?* (Oxford University Press, New York, 1941).

2. H.S.M. Coxter, *Introduction to Geometry* (Wiley, New York, 1963).

3. E.N. Gilbert and H.O. Pollak, Steiner minimal trees, *SIAM J. Appl. Math.* 16 (1968), 1.

4. C. Isenberg, *The Science of Soap Films and Soap Bubbles* (Tieto, Clevedon, 1978).