Parabola Volume 20, Issue 2 (1984)

THE 1984 SCHOOL MATHEMATICS COMPETITION

JUNIOR DIVISION

1. A diagonal of a polygon is a line joining two non-adjacent vertices. How many vertices has a ploygon which has 152 diagonals?

SOLUTION

If a polygon has n vertices then each vertex is joined to $n-3$ vertices by diagonals. Thus the number of diagonals = $n(n-3)/2$.

Hence
$$n(n-3)/2 = 152$$
$$n^2 - 3n - 304 = 0$$
$$(n-19)(n+16) = 0$$
and $n = 19$ since $n > 0$

2. Given that $3a^2 + 4b^2 + 18c^2 - 4ab - 12ac = 0$ prove that $a = 2b = 3c$.

SOLUTION

The only fact one can use to solve such an equation is
$$x^2 + y^2 + z^2 + \ldots = 0 \text{ implies } x = y = z = \ldots = 0.$$

Now $4b^2 - 4ab = (2b-a)^2 - a^2$ and $18c^2 - 12ac = 2(3c-a)^2 - 2a^2$

hence $3a^2 + 4b^2 + 18c^2 - 4ab - 12ac = 0$

implies $(2b-a)^2 - a^2 + 2(3c-a)^2 - 2a^2 + 3a^2 = 0$

and $(2b-a)^2 + 2(3c-a)^2 = 0$

hence $2b - a = 0$ and $3c - a = 0$

or $a = 2b = 3c$.

3. Consider the sequence $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 7, \ldots$ in which $a_{n+2} = a_{n+1}.a_n + 1$ for n greater than or equal to 1. Prove that no term is divisible by 4.

SOLUTION

Let b_i be the remainder on dividing a_i by 4. Then $b_1 = 1$, $b_2 = 1$, $b_3 = 2$, $b_4 = 3$, $b_5 = 3$, $b_6 = 3.3+1 \pmod 4 = 2$, $b_7 = 3.2+1 \pmod 4 = 3$, $b_8 = 2.3+1 \pmod 4 = 3$, $b_9 = 3.3+1 \pmod 4 = 2$, Clearly $\{b_i\} = 1, 1, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, \ldots$.

Thus $b_i \neq 0$ and no a_i is divisible by 4.

4. Is there a positive integer which reduces to its

a) 1/57th b) 1/56th part

when its first digit is deleted.

SOLUTION

(a) Represent any integer as $a.10^k + b$ with $1 \le a \le 9$ and $0 \le b \le 10^{k-1}$.

If this integer has the required property then

$$a.10^k + b = 57b$$
$$a.10^k = 56b$$

thus $a = 7$ and $b = 125.10^{k-3}$ hence solutions are 7125,71250,712500,..

(b) Similarly

$$a.10^k + b = 56b$$
$$a.10^k = 55b$$

has no solutions since $11|a$ and $1 \le a \le 9$.

5. (i) Show $(2n+1)^2 = (2n^2 + 2n + 1)^2 - (2n^2 + 2n)^2$

(ii) Construct an infinite sequence of distinct squares such that for all k, the sum of the first k squares is also a square.

SOLUTION

(i) $a^2 - b^2 = (a-b)(a+b)$ hence $(2n^2 + 2n + 1)^2 - (2n^2 + 2n)$

$= (2n^2 + 2n + 1 - 2n^2 - 2n)(2n^2 + 2n + 1 + 2n^2 + 2n) = 4n^2 + 4n + 1$

$= (2n + 1)^2$

(ii) $(2n + 1)^2 + (2n^2 + 2n)^2 = (2n^2 + 2n + 1)^2$

For example with $n = 1$ $3^2 + 4^2 = 5^2$

with $n = 2$ $5^2 + 12^2 = 13^2$

$n = 6$ $13^2 + 84^2 = 85^2$

$n = 42$ $85^2 + 3612^2 = 3613^2$

Clearly the sequence $3^2, 4^2, 12^2, 84^2, 3612^2, \ldots = a_1^{\,2}, a_2^{\,2}, a_3^{\,2}, \ldots$ has the property, where

$$a_{n+1} = \tfrac{1}{2}a_n^{\,2} + a_n \quad , \quad n \geq 2.$$

6. A, B, C are given numbers. Show that it is possible to find a positive integer N such that

$$A + Bn + Cn^2 < n!$$

for all integers $n > N$. $(n! = 1.2.\ldots.n)$.

SOLUTIONS

$6! = 720$, $10! = 3628800$ and $50! \cong 10^{64}$. Clearly the result is true, all that is needed is a rigorous argument. Here is one of many:

Let $N = \max(10, |A|, |B|, |C|)$ where $|x| = -x$ if $x < 0$ and $|x| = x$ if $x \geq 0$

then $A + Bn + Cn^2 < n + n^2 + n^3 < n(n+1)(n+2)$

if $n > N$ $\qquad < n(n-1)(n-2)(n-3)(n-4)$

$\qquad < n!$ since $N \geq 10$.

7. Albert, Bertram, Carol and Denise, whose surnames in some order are Edwards, Ford, Grant and Hanks, are in forms 3, 4, 5 and 6 at their high School. Their English marks in the recent examination were 55, 60 , 65 and 70. From the following clues match up forms, marks, and names.

1. Grant is in a higher form than Hanks.
2. Together the boys earned more marks for English than the girls.
3. The third-former had the highest English mark.
4. Albert and Hanks, comparing marks with a third member of the four pointed out that if they multiplied their form number by their English marks, they both beat her score of 260.
5. Bertram, drew with Edwards in the chess championship, but Denise beat Hanks.

SOLUTION

Since the boys earned more for English than the girls, one of them has 70 marks. This boy is a 3rd former. The girl whose score is 260 must be in 4th form with an English score of 65. Hanks must be in form 5 since he is in a lower form than Grant but a higher form than the 4th form girl. Hence Grant is in form 6, and is called Albert. Thus the third former is called Bertram. The rest follows easily.

Form 3	Bertram Ford	70
Form 4	Denise Edwards	65
Form 5	Carol Hanks	55
Form 6	Albert Grant	60

SENIOR DIVISION

1. Prove $(2 + \frac{10}{9}\sqrt{3})^{1/3} + (2 - \frac{10}{9}\sqrt{3})^{1/3} = 2.$ *(Do not use your calculator!)*

SOLUTION

Let $a = 2 + \frac{10}{9}\sqrt{3}$, $b = 2 - \frac{10}{9}\sqrt{3}$ then $a + b = 4$ and

$$ab = 4 - \frac{100}{81}.3 = \frac{8}{27}$$

Let $c = \sqrt[3]{a}$ and $d = \sqrt[3]{b}$ then $c^3d^3 = \frac{8}{27}$, $cd = \frac{2}{3}$

Also $c^3 + d^3 = 4$

$$(c + d)(c^2 - cd + d^2) = 4$$
$$(c + d)((c + d)^2 - 3cd) = 4$$
$$(c + d)^3 - 2(c + d) - 4 = 0$$
$$((c + d) - 2)((c + d)^2 + 2(c + d) + 2) = 0$$
$$c + d = 2 \quad \text{or} \quad c + d = (-2 \pm \sqrt{4-8})/2$$

hence

$$c + d = 2 \quad \text{or} \quad (2 + \frac{10}{9}\sqrt{3})^{1/3} + (2 - \frac{10}{9}\sqrt{3})^{1/3} = 2.$$

ALTERNATIVELY

Let $a = (2 + \frac{10}{9}\sqrt{3})^{1/3} + (2 - \frac{10}{9}\sqrt{3})^{1/3} = c^{1/3} + d^{1/3}$ say then $c + d = 4$

and $cd = 4 - \frac{100}{81}.3 = 8/27$. Therefore $(cd)^{1/3} = 2/3$

$$a^3 = (c^{1/3} + d^{1/3})^3 = c + 3c^{2/3}d^{1/3} + 3c^{1/3}d^{2/3} + d$$

$$= c + d + 3(c^{1/3} + d^{1/3})(cd)^{1/3}$$

$$= 4 + 3a.\frac{2}{3} = 4 + 2a$$

Hence $a^3 - 2a - 4 = 0$, $(a-2)(a^2 + 2a + 2) = 0$

Therefore $a = 2$.

2. Find the smallest value of $|36^m - 5^n|$ if m and n are positive integers. Prove your result.

SOLUTION

$36^m = 36, 1296, 46656, \ldots$ $5^n = 5, 25, 125, 625, 3125, \ldots$

The answer appears to be 11.

Now $36^m \equiv 6 \pmod{10}$, $5^n \equiv 5 \pmod{10}$

Hence $|36^m - 5^n| = 1, 9, 11, 19, \ldots$ etc.

However $36^m \equiv 0 \pmod 9$ and $5^n \not\equiv 0 \pmod 9$

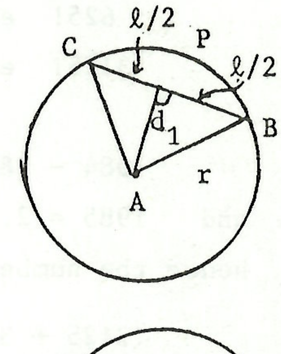
Thus $|36^m - 5^n| \neq 9$. If $36^m = 5^n + 1$ then $36^m \equiv 0 \pmod 4$ and $5^n + 1 \equiv 2 \pmod 4$. If $36^m - 1 = 5^n$ then $36^m \equiv 1 \pmod 7$ and $5^n \not\equiv 0 \pmod 7$. Thus the minimum possible value is 11 as expected.

3.(i) A point P is chosen at random on a circular disc. What is the chance that the chord with P as mid-point is longer than the radius?

(ii) Diameters AB, CD of a circle are chosen at random. What is the chance that both AC, AD are longer than the radius?

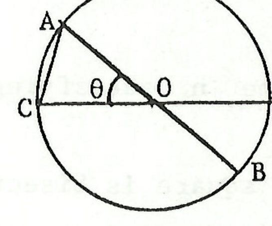
SOLUTION

(i) Clearly $\ell = r$ if and only if ABC is an equilateral triangle, in which case $d_1 = \sqrt{3}r/2$. Hence $\ell > r$ if and only if $d < \sqrt{3}r/2$.



The chance of this occuring

$$= \frac{\text{Area of circle radius AP}}{\text{Area of whole circle}} = \frac{\pi d_1^2}{\pi r^2} = \frac{3}{4}$$



(ii) Clearly $AC = AO \Leftrightarrow \theta = \pi/3$

$AD = AO \Leftrightarrow \theta = 2\pi/3$

Thus $AC > r$ and $AD > r$

$\Leftrightarrow \pi/3 < \theta < 2\pi/3$.

The chance of this occuring

$= \pi/3 \big/ \pi = 1/3$ ($\Leftrightarrow$ denotes if and only if)

4.(i) Prove that there is no whole number n such that $n!$ ends with exactly 1984 zeros.

(ii) Find those n for which $n!$ ends in exactly 1985 zeros.

SOLUTION

$n! = 2^a 5^b \ldots$ with $a > b$, hence the number of zeros depends on the power of 5 dividing $n!$

If we consider the sequences $1!, 2!, \ldots, k!, \ldots$. the number of zeros increases by

$$\begin{array}{ll} 1 & \text{if } 5|k \quad 25\nmid k \\ 2 & \text{if } 25|k \quad 125\nmid k \\ 3 & \text{if } 125|k \quad 625\nmid k \end{array}$$

Hence the number of zeros increases by

1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 3, ... at k =

5, 10, 15, 20, 25, 30, 35, 40,............, 120, 125, ...

with the first 3 occuring at 125.

and the first 4 occuring at 625 etc.

Hence 5! ends in 1 zero

25! ends in 6 zeros

125! ends in 31 zeros

625! ends in 156 zeros

3125! ends in 781 zeros.

$$1984 - 781 - 781 - 156 - 156 - 31 - 31 - 31 - 6 - 6 = 5$$

and $1985 = 2.781 + 2.156 + 3.31 + 3.6$

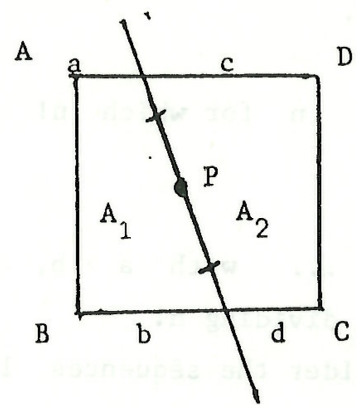
hence the number of zeros jumps from 1983 to 1985 at

$$(3125 + 3125 + 625 + 625 + 375 + 75)! = 7950!$$

The n satisfying condition (ii) are 7950, 7951, 7952, 7953 and 7954.

5. A square is bisected by each of nine lines into two quadrilaterals so that the area of one of these quadrilaterals is twice the area of the other. Prove that there is a point which lies on at least three of these lines.

SOLUTION



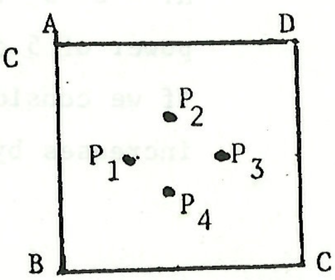
Area A_2 = 2. Area A_1

if and only if

$$\frac{a+b}{2} = \frac{1}{2}\left(\frac{c+d}{2}\right)$$

if and only if

P is 1/3 of the distance from AB to CD



Hence any line bisecting ABCD in the ratio 2:1 must pass through one of P_1, P_2, P_3 or P_4. Hence if 9 lines have this property at least 3 must pass through one of P_1, P_2, P_3 or P_4.

6. Prove that the sum of the reciprocals of all the positive integers which do not have the digit 0 in their usual expression in base 10 does not exceed 100.

SOLUTION

Let S be the set of positive integers with no 0 digit. There are 9 one digit numbers in S, 81 two digit numbers in S, 9^3 three digit numbers in S.

Thus $\sum_{n \in S} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{9} + \frac{1}{11} + \dots \quad < 9\,\frac{1}{1} + 9^2\,\frac{1}{10} + 9^3\,\frac{1}{100} + 9^4\,\frac{1}{1000} + \dots$

$$= \frac{9}{1 - \frac{9}{10}} = 90, \quad \text{as required.}$$

7. The sequence $a_1, a_2, \dots$ is defined by $a_1 = 1$,

$$a_{n+1} = \frac{1}{16}(1 + 4a_n + \sqrt{1 + 24a_n}) \text{ for } n > 1.$$

Prove that each member of this sequence is a rational number.

SOLUTION

$$1 + 24a_{n+1} = 1 + \frac{3}{2} + 6a_n + \frac{3}{2}\sqrt{1 + 24a_n}$$

$$= \frac{9}{4} + \frac{3}{2}\sqrt{1 + 24a_n} + \tfrac{1}{4}(1 + 24a_n)$$

$$= (3/2 + \tfrac{1}{2}\sqrt{1 + 24a_n})^2$$

Thus $\sqrt{1 + 24a_{n+1}} = 3/2 + \tfrac{1}{2}\sqrt{1 + 24a_n}$

Now a_1 is rational. Assume a_n is rational, hence $\sqrt{1 + 24a_{n-1}}$ is rational, thus $\sqrt{1 + 24a_n}$ is rational and hence by induction a_{n+1} is rational for all n.