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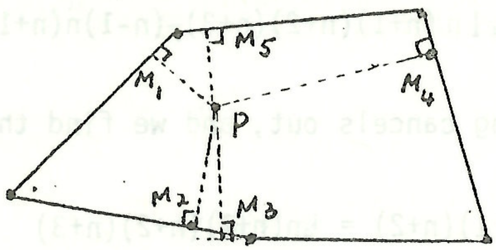
PROBLEM SECTION

Of the following set of problems, Q. 605 has appeared in the Wisconsin Mathematical Engineering and Science Talent Search, and Q. 600, Q. 601 and Q. 606 have been modified from the same source. Q. 602, 603, 604 and 610 have appeared in a Russian school magazine. Dr. P. Donovan suggested Q. 611.

Q.600. P is a point inside a convex polygon all of whose sides are of equal length. Perpendiculars are constructed from P to the sides of the polygon (produced if necessary). Show that the sum of the lengths of the perpendiculars ($PM_1 + PM_2 + \ldots + PM_5$ in the figure) is the same for all positions of P.



Q.601. Using your calculator you will be able to check that

$$\sqrt{5+\sqrt{21}} + \sqrt{8+\sqrt{55}} \quad \text{and} \quad \sqrt{7+\sqrt{33}} + \sqrt{6+\sqrt{35}}$$

are approximately equal. Either prove that they are exactly equal, or decide (with proof) which is the larger.

Q.602 ABCD is any rectangle, and ABX, BCY, CDV, and DAW are outward drawn equilateral triangles. Prove that the sum of the areas of the triangle AXW, BYX, CVY and DWV equals the area of the rectangle.

Q.603 Show that there is no party of 10 members in which the members have 9, 9, 9, 8, 8, 8, 7, 6, 4, 4, acquaintances among themselves. (Acquaintances may be assumed to be mutual).

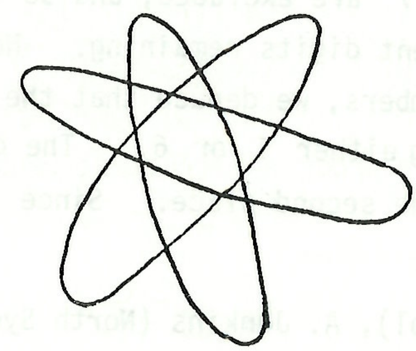
Q.604 Given a convex quadilateral ABCD show how to construct a point P in its interior such that the lines joining P to the mid points of the sides divide the quadrilateral into 4 equal areas.

Q.605 If a, b, c, d are four positive integers such that $ab = cd$, prove that $a + b + c + d$ is not a prime number.

Q.606 Suppose we start adding $1 + 1/4 + 1/9 + 1/16 + 1/25 + \ldots$ where the n th term is $1/n^2$. The more terms added, the closer the total gets to $\pi^2/6$. (This requires advanced methods to prove; the result was obtained by L. Euler). Show that if only k terms are added, the total differs from the eventual limit sum by less than $1/k$, where k is any positive integer. i.e. show that

$$\frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \frac{1}{(k+3)^2} + \ldots\ldots < 1/k,$$ however many terms on the left hand side are added.

Q.607 A number of ellipses are drawn in the plane, any two of them intersecting in 4 points. No three of the curves are concurrent. In the figure, three such ellipses have been drawn, and the plane has been divided by them into 14 regions (including the unbounded region lying outside all the ellipses). Into how many regions would the plane be divided if 10 ellipses were drawn.



Q.608 Find all integers x such that

$$x^4 + 4x^3 + 6x^2 + 4x + 5$$

is a prime number.

Q.609 Let x denote the number $11111111 \left(= \frac{10^8-1}{9} \right)$. If y is any multiple of x, show that the sum of the digits of y is not equal to 63. Find a multiple of x the sum of whose digits is 65.

Q.610 Find all triples of positive integers (x, y, z) such that $3^x + 4^y = 5^z$.

Q.611 Jim wins a secret ballot election with 7 votes, his only opponent Bill getting only 3 votes. Find the probability that as the votes were counted, Jim's tally remained greater than Bill's right from the first vote.