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SOLUTIONS TO PROBLEMS FROM VOLUME 19, NUMBER 3

Q.576. The game of Yellow Pigs is a favourite pastime at Hampshire College's Summer Science Training Program. It is played as follows. One player chooses three different numbers from the first seven positive integers to form a three digit number, and a second player tries to guess the chosen number by testing different three digit numbers in succession. Each guess is given an answer in the form "k pigs, n of them are yellow" meaning that k of his three digits were correct, and of these, n occupied the same position as in the chosen number. In one such game each of the first four guesses; 631, 237, 253, and 425 were answered by "1 pig; it is not yellow". What was the chosen number?

Solution: The digit 3 cannot be used, since then all of the digits 1, 2, 5, 6, and 7 would be excluded, leaving only two different digits available. Similarly, if 2 were used, all of 3, 4, 5, and 7 are excluded, and so is either 1 or 6; again there would be insufficient digits remaining. Now, referring to the second and third of the given numbers, we deduce that the digits 5 and 7 are both used, the remaining digit being either 1 or 6. The digit 5 must occupy the first place, and 7 must be in the second place. Since 1 is not in the last place, the number must be 576.

Correct solutions from: B. Coles (Nepean High School), A. Jenkins (North Sydney Boys' High School), L.A. Koe (James Ruse Agricultural High School), M. Leeming (Sydney Grammar School), J Percival (Elderslie High School).

Q.577. ABC is a triangle of area 1 unit. Points X,Y,Z lie on BC (produced), CA (produced) and AB (produced), so that BX = 3 times BC, CY = 3 times CA, and AZ = 3 times AB. Find the area of ΔXYZ.

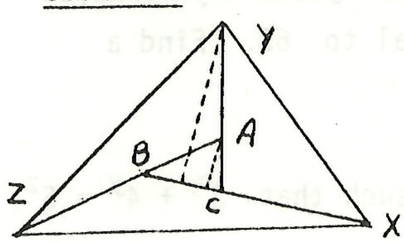
Solution: Area ΔYCX = ½ × base × height

= ½ × CX × height from Y to BX

= ½ × 2BC × 3 × height from A to BC

= 6 × Area ΔABC

= 6



(continued over)

Similarly, Area ΔAYZ = Area $\Delta ZBX = 6$. Hence, area $\Delta XYZ = 3 \times 6 + 1$
$= 19$ units.

Correct solution from: J. Percival (Elderslie High School).

Q.578. I have two perfect squares whose product exceeds their sum by 4844. Find them.

Solution:

$$x^2y^2 - (x^2 + y^2) = 4844$$

$$\Rightarrow \quad x^2y^2 - x^2 - y^2 + 1 = 4845$$

$$\Rightarrow \quad (x^2 - 1)(y^2 - 1) = 3 \times 5 \times 17 \times 19$$

$$\Rightarrow \quad (x - 1)(x + 1)(y - 1)(y + 1) = 3 \times 5 \times 17 \times 19.$$

Hence x and y must be 4 and 18 in either order. The squares are 16 and 324.

Correct solutions from: A. Jenkins (North Sydney Boys' High School), L.A. Koe (James Ruse Agricultural High School), J. Percival (Elderslie High School), I. Liubarsky (Moriah College).

Q.579. A set A contains n distinct numbers. The set S is constructed from A by the definition $S = \{x + y : x, y \in A\}$. i.e. the numbers in S are obtained by adding two numbers in A. (Note that x and y may be the same element of A.) Let m be the number of elements in the set S. Find the smallest and the largest possible values of m.

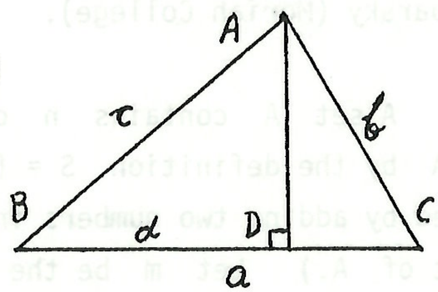
Solution: Let the numbers in A in increasing order be $a_1, a_2, a_3, \ldots, a_n$. Then $a_1 + a_1,\ a_1 + a_2,\ a_1 + a_3, \ldots, a_1 + a_n,\ a_2 + a_n,\ a_3 + a_n, \ldots, a_n + a_n$ are $2n - 1$ elements of S each larger than the preceding. Thus the smallest value of m is at least $2n - 1$. Taking any set of consecutive integers for $a_1, a_2, \ldots, a_n$, all elements of S are integers between $2a_1$ and $2a_n$ inclusive. Since there are only $2n - 1$ integers in that range, the minimum value $m = 2n - 1$ can be achieved.

The largest possible value of m would occur when all the sums $a_i + a_j$ give different answers (except, of course, that $a_i + a_j = a_j + a_i$ always). Hence an upper bound for m is ${}^nC_2 + n = \frac{1}{2} \times n \times (n - 1) + n$. (The second term counts the elements $a_1 + a_1$, $a_2 + a_2, \ldots, a_n + a_n$). It is easy to achieve this maximum value for m; for example, by taking $a_1 \geqslant 0$, $a_i \geqslant 2a_{i-1} + 1$ for $i = 2, 3, \ldots, n$.

Q.580. The lengths of the sides of a triangle ABC are all rational numbers. Let D be the foot of the perpendicular from A to BC. Show that the length of BD is a rational number.

Solution: Referring to the figure, and using the cosine rule $d = c \times \cos B$

$$= c \times \frac{(a^2 + c^2 - b^2)}{2.a.c}$$



The result follows at once, since sums, differences, products, and quotients of rational numbers are rational.

Correct solution from: J. Percival (Elderslie High School).

Q.581. The symbol $[x]$ denotes the greatest integer less than or equal to x. Find the complement of the set $E = \{[n + \sqrt{n + \frac{1}{2}}] : n \in \mathbb{N}\}$ in the set $\mathbb{N}$ of natural numbers. (i.e. Describe all whole numbers m such that m is not equal to $[n + \sqrt{n + \frac{1}{2}}]$ for any whole number n.)

Solution: As n takes in succession the values

$$(k - 1)^2, (k - 1)^2 + 1, (k - 1)^2 + 2, \ldots, k^2 - 1,$$

the value of $\sqrt{(n + \frac{1}{2})}$ lies between $(k - 1)$ and k, and the value of $[n + \sqrt{(n + \frac{1}{2})}]$ is equal to $n + (k - 1)$, which takes in succession the values from $(k - 1)^2 + (k - 1)$ to $k^2 + k - 2$. However, when n increases to k^2, $[(n + \frac{1}{2})]$ goes up to k, and the value k^2 is obtained for the given expression. Thus the numbers $k^2 + k - 1$, $k = 1, 2, 3, \ldots,$ are not in E, but all other non-negative integers are in E.

Correct solutions from: J. Percival (Elderslie High School), M. Leeming (Sydney Grammar School), B. Coles (Nepean High School).

Q.582. In the arithemtic progression

$$a,\ a + d,\ a + 2d,\ a + 3d, \ldots, a + nd, \ldots$$

three consecutive terms are perfect squares. Prove that d is a multiple of 24.

Solution: One can assume without loss of generality that a, $a + d$, and $a + 2d$ are the three squares. Since odd squares exceed by 1 a multiple of 8 (Proof:- $(2n + 1)^2 = 8 \times (n \times (n + 1)/2) + 1)$, and even squares are multiples of 4, it is easy to check that if r^2, s^2, t^2 are in arithmetic progression they must all be even, or else all odd. If all even, we can cancel out the common factor 4 to obtain a smaller set of squares $a/4$, $a/4 + d/4$, $a/4 + 2d/4$; we would eventually be able to conclude that $d/4$ is a multiple of 24, and d a multiple of 96. So it suffices to confine our attention to the case when a, $a + d$, and $a + 2d$ are all odd perfect squares. Since, as shown above, all odd squares exceed by 1 a multiple of 8, we see immediately that d must be a multiple of 8.

If r is a multiple of 3, so is r^2. Otherwise r differs from a multiple of 3 by 1, and r^2 is one more than a multiple of 3. (Proof:- $(3n \pm 1)^2 = 3 \times (3n^2 \pm 2n) + 1)$. Thus r^2, s^2, t^2 are in arithmetic progression only if all three squares are multiples of 3, or else all three are one more than multiples of 3. In either case, the difference, d, is a multiple of 3. Since 3 and 8 are both factors of d, clearly d is a multiple of 24.

Q.583. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:

i) $f(xf(y)) = yf(x)$ for all positive x, y

ii) $f(x) \to 0$ as $x \to +\infty$.

Solution: Call t a fixed point of f if $f(t) = t$. Setting $y = x$ in i) shows that $xf(x)$ is a fixed point of f for any x, so f certainly has at least one fixed point. Note that if t is a fixed point, so are $t^2, t^3, \ldots, t^n, \ldots$. Indeed, setting $x = t^{n-1}$, $y = t$ in i), and assuming that t^{n-1} is a fixed point, we obtain $f(t^n) = f(tf(t^{n-1})) = f(t)t^{n-1} = t^n$.

Again, setting $x = 1/t$, $y = t^2$ we get $t = f(t) = f(f(t^2)/t) = t^2f(1/t)$. Dividing by t^2, we obtain $f(1/t) = 1/t$, so that $1/t$ and all its powers must also be fixed points of f.

We can now prove using ii) that f has only one fixed point, namely 1. For if t is a fixed point not equal to 1, we can find an unbounded increasing sequence of fixed points (either t^n or $1/t^n$) and this contradicts the requirement $f(x) \to 0$ as $x \to \infty$.

Hence, for every x, the fixed point $xf(x)$ must be equal to 1, and it follows that the only such function is defined by $f(x) = 1/x$.

Q.584. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centres O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that the angles O_1AO_2 and M_1AM_2 are equal.

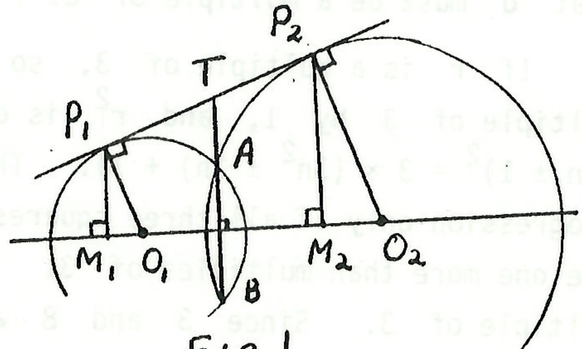


Fig 1

Solution: Refer to Figure 1. Let the common chord BA of the two circles meet P_1P_2 at T. Since $TP_1^2 = TA.TB = TP_2^2$, the parallel lines P_1M_1, TA, and P_2M_2 are equidistant, so TA is the perpendicular bisector of M_1M_2, and $AM_1 = AM_2$. Thus

$$\angle AM_1M_2 = \angle AM_2M_1 \tag{1}$$

From the similar triangles $P_1M_1O_1$ and $P_2M_2O_2$ we see at once that

$$O_1M_1 : O_2M_2 = r_1 : r_2 \tag{2}$$

where r_1, r_2 are the lengths of the radii of C_1, C_2 respectively.

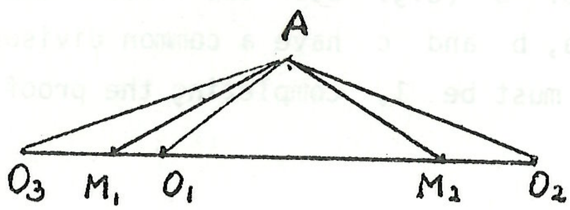


Fig 2

In Figure 2, the point O_3 on O_2O_1 produced is such that $M_1O_3 = M_2O_2$.

It is easy to check, using (1), that $\Delta AO_3M_1 \equiv \Delta AO_2M_2$, as the triangles are "mirror images" of each other in the line AB. Thus $AO_3 = r_2$, and $\angle O_3AM_1 = \angle O_2AM_2$. Now in ΔAO_1O_3, since $O_1M_1/M_1O_3 = O_1M_1/M_2O_2 = r_1/r_2 = O_1A/O_3A$, it follows that AM is the angle bisector of $\angle O_1AO_3$. Hence $\angle O_1AM_1 = \angle M_1AO_3 = \angle M_2AO_2$. The required result follows immediately from this.

<u>Q.585</u>. Let a, b, c be positive integers, no two of which have a common divisor greater than 1. Show that

$$2abc - ab - bc - ca$$

is the largest integer which cannot be expressed in the form

$$xbc + yca + zab$$

where x, y, z are non-negative integers.

<u>Solution</u>: If x, y, z may be <u>any</u> integers (negative or non-negative) then any integer m may be expressed in the form

$$m = xbc + yca + zab.$$

(continued over)

[Proof: Let S denote the collection of all integers so expressible, and let d be the smallest positive member of S; $d = x_1bc + y_1ca + z_1ab$. Let u be any other element of S, and let q, r be the quotient and remainder when u is divided by d, i.e. $u = qd + r$ with $0 \leqslant r < d$. Since $r = u - (qx_1bc + qy_1ca + qz_1ab)$ is the difference of 2 elements of S it is itself an element of S, smaller than d. By the definition of d, r cannot be positive, so we must have $r = 0$. Thus every element of S is a multiple of d, and it follows that S consists precisely of all multiples of d. In particular bc, ca, and ab, being elements of S (e.g. $bc = 1bc + 0ca + 0ab$) are multiples of d. But since no two of a, b and c have a common divisor greater than 1, it easily follows that d must be 1, completing the proof of the assertion.]

Suppose

$$xbc + yca + zab = x'bc + y'ca + z'ab.$$

Then $(x - x')bc + (y - y')ca + (z - z')ab = 0$. Since a is a factor of the last 2 terms, we deduce a divides $(x - x')bc$, and therefore a divides $x - x'$. Similarly $b|(y - y')$ and $c|(z - z')$. If $x - x' = \lambda a$ and $y - y' = \mu b$ then $z - z' = -(\lambda + \mu)c$. Now the given number $2abc - bc - ca - ab$, has one expression in the desired form with $x = 2a - 1$, $y = -1$, $z = -1$. Other possible expressions will have

$$x' = 2a - 1 + \lambda a, \quad y' = -1 + \mu b, \quad z' = -1 - (\lambda + \mu)c$$

where λ, μ are integers.

In order to have $x' \geqslant 0$ we must have $\lambda \geqslant -1$
$y' \geqslant 0$ we must have $\mu \geqslant +1$.

But then $\lambda + \mu \geqslant 0$ and it is clear that $z' < 0$. Thus there is no expression for the given number in which all of x, y, and z are positive.

However, if

$$\begin{aligned} 2abc - bc - ca - ab < m &= x_0bc + y_0ca + z_0ab \\ &= (x_0 + \lambda a)bc + (y_0 + \mu b)ca + (z_0 - (\lambda + \mu)c)ab \end{aligned}$$

we can choose λ, μ so that $0 \leqslant x' = x_0 + \lambda a < a$ and $0 \leqslant y' = y_0 + \mu b < b$.

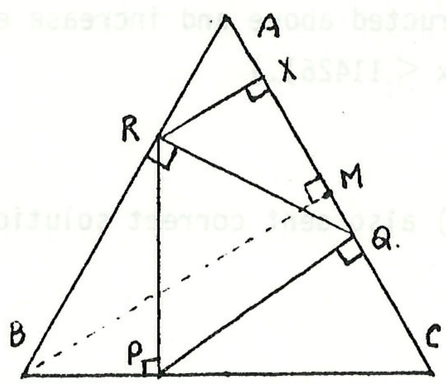
Then $z'ab = m - x'bc - y'ca > (2abc - bc - ca - ab) - ((a - 1)bc + (b - 1)ca)$

$$= - ab$$

Therefore $z' > - 1$ so $z' \geqslant 0$.

Hence an expression $m = x'bc + y'ca + z'ab$ exists with all of x', y', z' non-negative integers. Q.E.D.

Q.586. Let ABC be an equilateral triangle, and E be the set of all points contained in the three segments AB, BC and CA (including A, B, and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.



Solution: If P, Q and R are points of trisection of the sides as shown in the figure, it is easy to check that $PQ \perp AC$, $QR \perp AB$ and $RP \perp BC$. Let E_1, E_2 be the sets of points constituting the partition of E. Either E_1 or E_2 must contain at least 2 of the three points {P, Q, R}. For definiteness, suppose P and Q are both in E_1 (this can always be achieved by appropriate renaming of points and subsets). To avoid a right angled triangle in E_1 every other point on AC must be in E2, and then R must be in E_1 to avoid a triangle RXA, say, in E_2. But now if B is in E_1, all vertices if the right angled triangle QRB are in E_1, and if B is in E_2 all vertices of BMC are in E_2. Thus at least one of the two subsets E_1, E_2 must contain the vertices of a right angled triangle.

Q.587. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.

Solution: Consider the set of positive whole numbers whose expressions in base 3 have no more than 11 digits, and the digit 2 does not occur. The largest of these is the number $11111111111_3 = 1 + 3 + 3^2 + \ldots + 3^{10} = \frac{3^{11} - 1}{3 - 1} = 88573$.

There are 2 ways of choosing the digit (0 or 1) for each of the eleven places so there are altogether $2^{11} - 1 = 2047$ such numbers. (The -1 is to exclude the number 0.) If any 1983 of these numbers are taken, we claim that no three of them a, b, c are consecutive turns of an AP, i.e. $a + c \neq 2b$. This is fairly obvious, since 2b consists of a number whose ternary representation contains only 0's and 2's. However if a and c are any 2 different members of the set, there is some ternary place in which the digits 0 and 1 occur in the two summands, and therefore the digit 1 must appear in the sum, as no "carrying" takes place when the addition is performed. (There are many other sets of 1983 numbers < 100000 without any A.P.'s. For example take one of the sets constructed above and increase every member by the same whole number x (where $0 < x < 11426$).)

Additional Solver

K. Boroczky. (St. Patrick's College, Strathfield) also sent correct solutions to problems 576, 577, 578, 580, 581.