MAXIMA & MINIMA WITH THE FIBONACCI NUMBERS: FIBONACCI SEARCH TECHNIQUES

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Introduction

Suppose you were required to find the minimum value of

$$f(x) = x^4 - 15x^3 + 72x^2 - 1135x \text{ for } 1 \le x \le 15.$$

If you use the calculus, you try,

$$f'(x) = 4x^3 - 45x^2 + 144x - 1135 = 0 \quad \text{for minimum } f(x),$$

but this looks tedious to attempt to solve.

We can use the *Fibonacci numbers* (cf. Cooper (1982)) to search for this minimum value as we now show. The general problem is optimise $f(x)$, a strictly unimodal function,

$$\text{subject to } a \le x \le b$$

where f is a function defined on the real line segment $[a,b]$.

In a search method we identify an *interval of uncertainty* which is known to include the optimum. We then systematically reduce the size of this interval to minimise the length which contains the optimum we seek.

Method

We use an inductive process to develop a plausible method, although it can be established more rigorously. In effect, we look for points of the interval where we evaluate our function until we get what seems

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the best result.

We let L_n be the length of the interval of uncertainty after n evaluations of the objective function $f(x)$ which will have a value at that point of

$$f_n = f(x_n) .$$

Clearly,

$$\begin{aligned} L_0 &= b - a \\ &= L_1 \end{aligned}$$

since a single functional evaluation at some arbitrary point x_1 of $[a,b]$ will not increase our certainty (Figure 1).

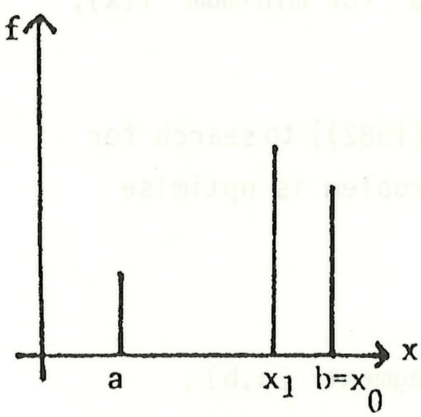

Figure 1

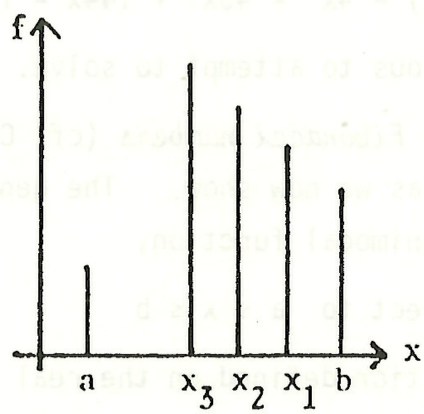

Figure 2.

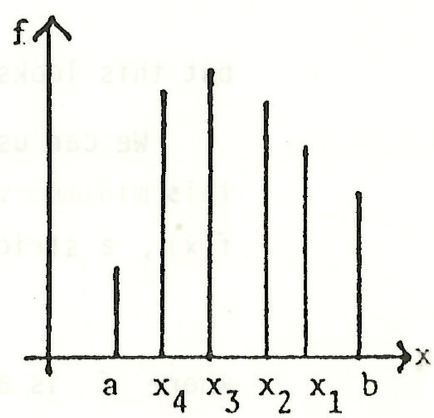

Figure 3

Suppose we seek maximum f for $x \in [a,b]$.
If $f(x_3) > f(x_2) > f(x_1)$ for our next two evaluations, then in turn

$$L_2 = x_1 - a$$

and $$L_3 = x_2 - a , \qquad \text{(Figure 2).}$$

Similarly, if $f(x_4) > f(x_3)$,

then

$$L_4 = x_3 - a .$$

Suppose though that

$$f(x_4) < f(x_3) \qquad \text{(Figure 3)}.$$

Then $\quad L_4 = x_2 - x_4 .$

We require that the uncertainty be the same in either case; that is,

$$x_3 - a = x_2 - x_4 .$$

This can be rewritten as

$$x_2 - a = (x_3 - a) + (x_4 - a),$$

or, more generally, as

(1) $$L_{n-j} = L_{n-j+1} + L_{n-j+2}, \quad j = 2, 3, \ldots .$$

We can also write

$$2L_4 = (x_2 - x_4) + (x_3 - a)$$

$$= (x_2 - a) + (x_3 - x_4),$$

or, again more generally, as

$$2L_n = (x_{n-2} - a) + (x_{n-1} - x_n)$$

or,

(2) $$2L_n = L_{n-1} + \varepsilon,$$

in which $\quad \varepsilon = x_{n-1} - x_n$

is the resolution factor.

If $j = 2$ in (1) we have the recurrence relation

(3) $$L_{n-2} = L_{n-1} + L_n = 3L_n - \varepsilon \qquad \text{(from (2))}.$$

For $j = 3$, $$L_{n-3} = L_{n-2} + L_{n-1} = 5L_n - 2\varepsilon;$$

for $j = 4$, $$L_{n-4} = L_{n-3} + L_{n-2} = 8L_n - 3\varepsilon,$$

or, in general,

(4) $$L_{n-j} = F_{j+2}L_n - F_j\,\varepsilon, \quad j = 1, 2, \ldots, n - 1,$$

where F_n is the nth Fibonacci number.

When $j = n - 1$, equation (4) becomes

$$L_1 = F_{n+1}L_n - F_{n-1}\varepsilon,$$

or $$L_n = \frac{L_1}{F_{n+1}} + \frac{F_{n-1}}{F_{n+1}}\varepsilon,$$

and $$\lim_{\substack{\varepsilon\to 0\\ n\to\infty}} L_n = \lim_{\substack{\varepsilon\to 0\\ n\to\infty}} \left(\frac{b - a}{F_{n+1}} + \frac{F_{n-1}}{F_{n+1}}\varepsilon\right) = 0$$ so that the algorithm will converge.

The *recurrence relation* (3) can be rewritten as

$$\frac{L_{n-2}}{L_{n-1}} = 1 + \frac{L_n}{L_{n-1}}$$

or for n *large enough*,

$$\frac{1}{m} = 1 + m \quad \text{where } 0 < m = L_n/L_{n-1} < 1 \text{ for convergence.}$$

That is, $$0 = m^2 + m - 1$$

and $$m = \tfrac{1}{2}(\sqrt{5} - 1)$$ $$= 0.618.$$

This is known as the *golden ratio* or *golden section*, and we use it as follows.

$$L_n = \frac{L_n}{L_{n-1}} \frac{L_{n-1}}{L_{n-2}} \cdots \frac{L_1}{L_0} (b - a)$$

$$= m^n(b - a)$$

$$= m(m^{n-1}(b - a))$$

$$= mL_{n-1}.$$

Example

Minimise $f(x) = x^4 - 15x^3 + 72x^2 - 1135x$ subject to $1 \leq x \leq 15$,

and terminate when $|f(x_n) - f(x_{n-1})| \leq 0.5$.

$$L_1 = 15 - 1 = 14$$

$$x_1 = 1 + L_1 m = 9.652$$

$$x_2 = 15 - L_1 m = 6.348$$

$$f(x_1) = 595.70 > -168.82 = f(x_2)$$

so the region to the right of x_1 can be eliminated (Figure 4).

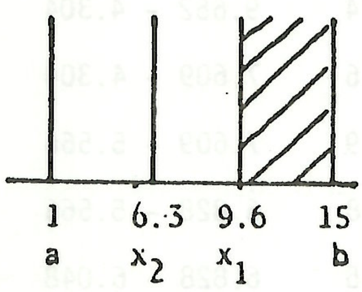

Figure 4

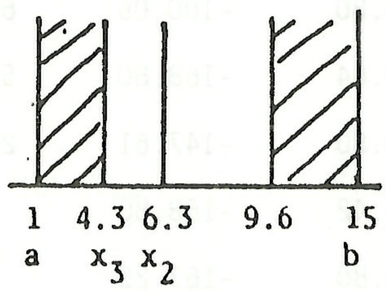

Figure 5

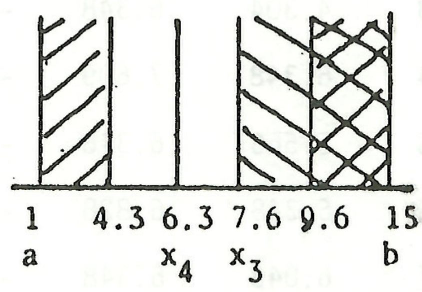

Figure 6

$$L_2 = 9.652 - 1 = 8.652$$
$$x_3 = x_1 - L_2 m = 4.304$$
$$f(x_2) = -168.82 < -100.06 = f(x_3)$$

so the region to the left of x_3 can be eliminated (Figure 5).

$$L_3 = 9.652 - 4.304 = 5.348$$

$$x_4 = 6.348$$

$$x_3 = 4.304 + L_3 m = 7.609$$

$$f(x_3) = -114.64 > -168.80 = f(x_4)$$

so the region to the right of x_3 can be eliminated, (Figure 6) and so on.

The iterations are displayed in Table 1, in which

$$d = |f(x_n) - f(x_{n-1})|.$$

Table One: Golden Search Iterations

n	x_n (left)	x_{n-1} (right)	$f(x_{n-1})$	$f(x_n)$	d	L_n
2	6.348	9.652	595.70	-168.80	764.50	9.652 - 1
3	4.304	6.348	-168.80	-100.06	68.74	9.652 - 4.304
4	6.348	7.609	-114.64	-168.80	53.36	7.609 - 4.304
5	5.566	6.348	-168.80	-147.61	21.19	7.609 - 5.566
6	6.348	6.828	-166.42	-168.80	2.38	6.828 - 5.566
7	6.048	6.348	-168.80	-163.25	5.55	6.828 - 6.048
8	6.348	6.530	-169.83	-168.80	1.03	6.828 - 6.346
9	6.530	6.643	-169.34	-169.83	0.49	6.643 - 6.346

Termination criteria are satisfied and so our best answer is given by

$$x^* = \tfrac{1}{2}(6.643 + 6.346) = 6.4945$$

so that

$$f(x^*) = -169.80.$$

Conclusion

The Fibonacci and Golden Section Search Algorithms are very effective in dealing with univariate nonlinear functions that are assumed to be unimodal. This assumption guarantees only one local optimum in the interval. Further information on the technique can be found in Wilde (1964).

Readers might like to show that the maximum value of

$$f(x) = -3x^2 + 21.6x + 1.0, \quad 0 \le x \le 25$$

is 39.83, and compare this result with the answer from the calculus method.

Reference

Cooper, Martin, "Fibonacci and Lucas Numbers", *Australian Mathematics Teacher*, 38, 2, 1982, 8-10.

Wilde, D. J., *Optimum Seeking Methods*, Prentice-Hall, Englewood Cliffs, 1964.

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