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MAKING SENSE OF NONSENSE

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1. Introduction

Mathematicians love formulae. Nothing pleases a mathematician more than to come up with a nice simple formula which solves a problem or summarises a result. Sometimes such formulae are true only under certain circumstances and when you go beyond those circumstances you obtain what appears to be nonsense results. For example early in our mathematical career we learn that we shouldn't take square roots of negative numbers; later however we're taught about complex numbers and what appeared to be nonsense is now made sensible. What follows is a further example of this phenomenon.

2. The Formula and the Nonsense

One of the more important formulae in mathematics is the formula for summing an infinite geometric progression

$$\sum_{n=0}^{\infty} X^n = 1 + X + X^2 + X^3 \dots = \frac{1}{1-X}$$

provided $|X| < 1$.

For those who might be unfamiliar with this, I shall include a brief sketch of its proof.

Let

$$S_n = 1 + X + X^2 + \dots + X^{n-1}$$

Then

$$X S_n = X + X^2 + X^3 + \dots + X^n$$

so that

$$S_n(1-X) = 1 - X^n$$

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or

$$S_n = \frac{1-X^n}{1-X} .$$

Thus

$$S_n - \frac{1}{1-X} = \frac{X^n}{X-1} .$$

Now for $|X| < 1$, $X^n \to 0$ as $n \to \infty$

so that "in the limit" $1 + X + X^2 + X^3 + \ldots = \frac{1}{1-X}$.

The important constraint for such a formula to be true is that $|X| < 1$. Thus if we substitute a value of X with $|X| > 1$ we will get nonsense. In fact if we put X = 2 we get

$$1 + 2 + 4 + 8 + 16 + \ldots = -1 \qquad (*).$$

This is the nonsense we shall seek to make sensible.

A Digression, Perhaps ?

Let's turn our mind to computers. It's well known that computers store numbers as strings of zeros and ones. This can be done by storing numbers in "binary" form, or in "base 2". For example in base 2,

$$13 \equiv 1101$$
$$\text{i.e.} \quad 13 = 1.2^3 + 1.2^2 + 0.2 + 1.$$

(This is just like decimal notation with 10 replaced by 2 !).
What then is the systematic procedure for writing a number in binary form?

Let

$$n = a_0 + a_1 2 + a_2 2^2 + a_3 2^3 + \ldots + a_k 2^k$$

with

$$a_i = 0 \text{ or } 1,$$

so that

$$n = a_k a_{k-1} \ldots . a_1 a_0$$

in binary form.

Put

$$n = n_0 ; \text{ if } 2|n_0 , a_0 = 0$$

otherwise

$$a_0 = 1.$$

Put

$$n_1 = \frac{n_0 - a_0}{2} ; \text{ if } 2|n_1 , a_1 = 0$$

otherwise

$$a_1 = 1.$$

Put

$$n_2 = \frac{n_1 - a_1}{2} ; \text{ if } 2|n_2 , a_2 = 0$$

otherwise

$$a_2 = 1 \quad \text{etc.}$$

This algorithm determines the values of a_i. Now let us try this algorithm for $n = -1$!

$n_0 = -1$ gives $a_0 = 1$ so that $n_1 = -1$ giving $a_1 = 1$

so that $n_2 = -1$ so that $a_2 = 1$ etc.

Thus for $n = -1$ we get

$$-1 = 1 + 2 + 4 + 8 + \ldots. \qquad (*)$$

This may not make sense of (*) but it does show we have to be careful representing negative numbers in binary notation! Strangely enough one way around this problem is to write numbers, not base 2, but base (-2) (!) i.e. in the form

$$n = a_0 + a_1(-2) + a_2(-2)^2 + \ldots \qquad + a_k(-2)^k$$

with $a_i = 0$ or 1 .

In fact every integer, positive and negative, can be represented as a <u>finite</u> sum of this type!

The more ambitious reader might like to try to prove this. I would suggest that they try to prove the following claim:

"If $0 \le n \le 2^{2k}$, then there exist $a_i = 0$ or 1 such that

$$n = a \;+ a\,(-2) + \dots \qquad + a_{2k}(-2)^{2k} \text{ "}$$

by using induction on k. A similar statement can be proved for negative integers.

<u>Sense at Last !</u>

As we noted our proof of the geometric progression formula worked provided $|X| < 1$. Thus what we really need to do is make sense of $|2| < 1$! To do this we define a different type of absolute value which we shall write as $|X|_2$

For X rational, we can write

$$X = 2^n \frac{a}{b}$$

where a, b are not divisible by 2. Define

$$|X|_2 = |2^n \tfrac{a}{b}|_2 = 2^{-n} .$$

This gives a new absolute value with properties very similar to the old one

1) $|XY|_2 = |X|_2 |Y|_2$.

2) $|X+Y|_2 \le |X|_2 + |Y|_2$ (triangle inequality) .

In fact 2) can be replaced by an even stronger statement

2') $|X+Y|_2 \le \max\{|X|_2, |Y|_2\}$.

The reader should easily be able to convince him or herself that 1) and 2') hold.

We can now make sense of (*) provided we are willing to introduce new types of numbers, numbers looking like

$$a_0 + a_1 2 + a_2 2^2 + \dots \qquad + a_n 2^n + \dots$$

with $a_i = 0$ or 1. In our new absolute value the "tail" of these numbers is not getting bigger and bigger, but smaller and smaller since

$$|2^n|_2 = 2^{-n}$$

For these new numbers

$$-1 = 1 + 2 + 4 + 8 \ldots$$

Similarly

$$-\frac{1}{3} = 1 + 4 + 16 + 64 \ldots$$

In fact there is nothing special about 2 in all this, any prime p will do. Define

$$|p^n \frac{a}{b}|_p = p^{-n}$$

where a, b are not divisible by p. Then

1) $|XY|_p = |X|_p |Y|_p$

2') $|X+Y|_p \quad \max \{|X|_p, |Y|_p\}$

and we can consider numbers which look alike

$$a_0 + a_1 p + a_2 p^2 + a_3 p^3 \ldots .$$

where $a_i \in \{0, 1, 2, \ldots, p-1\}$. For a fixed p such numbers can be added and multiplied in a fiarly obvious fashion. In fact we can even make sense of something like

$$\sqrt{-1} = 2 + 1.5 + 2.5^2 + 1.5^3 + \ldots$$

(the reader might like to calculate the next few terms !) .

It can be shown that the numbers we have introduced make as much sense as introducing complex numbers in order to allow the square roots of negative numbers. To justify that however would take us beyond the limited goal I have set myself.

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