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WOULD YOU BELIEVE IT?

BY

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These are not tall stories. All the assertions are true, though you may think some of them are very hard to believe at first. The arguments range in diff culty and sophistication very considerably; and all that the results have in common is that most of us would find them to be surprising.

1. A buckled railway line

We begin with a very elementary problem. Figure 1 illustrates a piece of railway line which was originally 1 mile long. It is buckled in the sun, and, as a result, expands

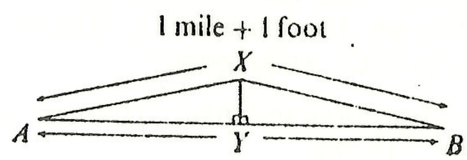

Figure 1

(uniformly) by 1 foot while the ends A, B remain fixed. For simplicity, we suppose that it becomes lifted from the ground at its middle point X as shown. At a guess, how high is X above the ground? Perhaps a few inches or maybe a yard? No, X is more than 50 feet above the ground!

If we measure in feet (remembering that 5280 feet = 1 mile), we have, in the notation of Figure 1,

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$$XY^2 = AX^2 - AY^2$$
$$= (2640 \cdot 5)^2 - 2640^2$$
$$= 2640 \cdot 25$$

and so

$$XY \simeq 51 \cdot 4 .$$

2. Common birthdays

This problem and the next one are probabilistic. To set the scene, suppose you were to ask two dozen people in the street (chosen at random) the dates of their birthdays. Then it is more likely than not that you would find that two of them had birthdays on the same day. The assertion that this is so for such a small number of people is usually greeted with surprise. So let us supply a proof. However, to begin with, we should emphasize that all we are saying is that the probability of the occurrence ot two birthdays falling on the same date exceeds ½. Without too much inaccuracy, we shall suppose that there are 365 days in each year and that all dates are equally likely to be birthdays. Then the number of lists of n birthdays is 365^n, the number containing no repeated date is

$$365(365 - 1)(365 - 2)\dots(365 - n + 1),$$

and so the probability of the occurrence of a repeated birthday for n people is

$$1 - \frac{365(365 - 1)(365 - 2)\dots(365 - n + 1)}{365^n}$$

$$= 1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\dots\left(1 - \frac{n-1}{365}\right) = 1 - p_n \quad \text{(say).}$$

Now

$$\log_e p_n = \sum_{k=1}^{n-1} \log_e \left(1 - \frac{k}{365}\right)$$

and, for n small compared with 365,

$$\log_e p_n \simeq \sum_{k=1}^{n-1} - \frac{k}{365}$$

$$= - \frac{1 + 2 + \ldots + n - 1}{365}$$

$$= - \frac{n(n-1)}{2 \times 365} .$$

Thus, for $n = 24$,

$$\log_e p_n \simeq - 0{\cdot}76$$

whereas

$$\log_e \tfrac{1}{2} \simeq - 0{\cdot}69 .$$

Therefore, for $n = 24$, it follows that $1 - p_n > \frac{1}{2}$. (The critical number is 23 in fact, not 24, but since our calculations are approximate we should need to discuss the size of the error term to convince ourselves of the stronger result.†)

3. The hats problem

Ten gentlemen attend a small party and leave their hats in the cloakroom. One thousand gentlemen attend a public reception and leave their hats in the cloadroom. On each occasion some confusion arises, and the hats are handed back to the guests at random. The probability, in each case, that no gentleman receives his own hat is about 0·37. Indeed, this probability to all intents and purposes is independent of the number of people involved.†† This comes as something of a surprise; but a few fairly simple calculations will convince us.

Consider n individuals (with n hats); and denote by D_n the number of ways in which they can receive the wrong hats. This is equal to the number of permutations $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$ such that $a_i \neq i$ for each i, i.e. the number of derangements of $(1, 2, \ldots, n)$. The probability that no man receives his own hat is then $D_n/n!$, which is what we wish to calculate.

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† In this connexion, we mention a recent article on the birthday problem by Susan Wilson (Mathematical Spectrum, Volume 13, Number 2), where a table of computed values for p_n is given.

†† This problem was also considered by Harris S. Schultz in Volume 12, Number 2 of Mathematical Spectrum.

To this end, consider those derangements of $(1,2,\ldots,n)$ in which the first position is occupied by the integer $k \neq 1$. The number of these in which the integer 1 occupies the kth position is evidently equal to D_{n-2}; and the number in which 1 does not occupy the kth position is D_{n-1} (since we may regard the kth position as a forbidden position not for k but for 1). Since k itself may take any one of the $n - 1$ values $2,3,\ldots n$, we obtain the relation

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

provided $n \geq 3$. Also $D_1 = 0$ and $D_2 = 1$. Thus

$$\begin{aligned} D_n - nD_{n-1} &= -(D_{n-1} - (n-1)D_{n-2}) \\ &= (-1)^2(D_{n-2} - (n-2)D_{n-3}) \\ &= \ldots \\ &= (-1)^{n-2}(D_2 - 2D_1) \\ &= (-1)^{n-2} \\ &= (-1)^n \, . \end{aligned}$$

It is helpful to write this last equation in the form

$$\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \, .$$

Now if we examine this for the first few values of n we shall see the pattern at once:

$$\frac{D_3}{3!} = \frac{D_2}{2!} + \frac{(-1)^3}{3!} = \frac{1}{2!} - \frac{1}{3!}$$

$$\frac{D_4}{4!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

and so

$$\frac{D_n}{n!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!}$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!} .$$

Further,

$$\left|\frac{D_n}{n!} - e^{-1}\right| = \frac{1}{(n+1)!} - \frac{1}{(n+2)!} + \frac{1}{(n+3)!} - \dots$$

$$= \frac{1}{(n+1)!} - \left(\frac{1}{(n+2)!} - \frac{1}{(n+3)!}\right) - \dots$$

$$< \frac{1}{(n+1)!}$$

and the right-hand side is small even for quite small values of n; for instance, for $n = 5$ it is equal to 1/720. Therefore $D_n/n! \simeq e^{-1} \approx 0{,}37$.

4. Infinite collections

It is not difficult to find very surprising results when we venture into the infinite. Let us begin our discussion by getting our basic notions clearly defined. Two finite sets evidently have the same number of elements in them precisely when they can be put in one-to-one correspondence with each other. It seems entirely reasonable, therefore, to take this as the definition of 'having the same number of elements' for sets which are not necessarily finite. When we do so, however, we meet with some surprises. For instance, in the collection of all (positive) integers, the even integers form a proper sub-collection; but, on the other hand, in view of the pairing

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & \dots \end{array}$$

we are bound to admit that, according to our definition, there are just as many even integers as there are integers altogether. This is only the first of the surprises that await us, however. Surely there are more fractions than there are integers! Let us consider the fraction p/q, where p and q are positive integers, and associate with it the symbol (p,q). The diagram

$$\begin{array}{ccccccc}
(1,1) & \rightarrow & (1,2) & & (1,3) & \rightarrow & (1,4) \ \ldots \\
 & & \downarrow & & \uparrow & & \downarrow \\
(2,1) & \leftarrow & (2,2) & & (2,3) & & (2,4) \ \ldots \\
\downarrow & & & & \uparrow & & \downarrow \\
(3,1) & \rightarrow & (3,2) & \rightarrow & (3,3) & & (3,4) \ \ldots \\
 & & & & & & \downarrow \\
(4,1) & \leftarrow & (4,2) & \leftarrow & (4,3) & \leftarrow & (4,4) \ \ldots
\end{array}$$

indicates just how we can make a list of all these, and thus pair them off with the integers 1,2,3,.... If we delete symbols which represent the same fraction (for instance (2,4), (3,6),..., which all represent the same fraction ½) we still get a list; namely $1, \frac{1}{2}, 2, 3, \frac{3}{2}, \frac{2}{3}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}$ and so on (in the ordinary notation for fractions). Therefore there are just as many integers as there are fractions. In contrast to all this, it is possible to show that the real numbers cannot be paired off with the integers, and thus that there are genuinely more real numbers than integers or fractions. Instead of looking into the proof of this last assertion, we turn to something a little different and very surprising. Let us consider the unit interval on a straight line. Its length is 1 unit. We shall proceed to show how to delete from it a sequence of subintervals whose lengths together add up to 1 unit and yet leave behind just as many points as there were in the original interval. It will make our calculations simpler (but is not significant in any other way) if we work with intervals which are 'closed' on the left and 'open' on the right, so that for instance our original interval includes 0 but does not include 1. We shall indicate this by writing this interval as

Figure 2

$[0,1)$; with a similar notation for the deleted subintervals. First, we delete the 'middle third' of $[0,1)$, namely $[\frac{1}{3},\frac{2}{3})$; next, the middle third of each of the remaining two intervals, namely $[\frac{1}{9}, \frac{2}{9})$ and $[\frac{7}{9}, \frac{8}{9})$; and so on, always removing middle thirds. Let us denote by F the set of points which ultimately remain. The total length removed is evidently equal to

$$\frac{1}{3} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{27} + \ldots$$

$$= \frac{1}{3} + \frac{2}{2^2} + \frac{2^2}{3^3} + \ldots$$

$$= \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1 .$$

Next, let us consider any point x of the original interval $[0,1)$ and write x in binary form as $0.b_1b_2b_3\ldots$. Each b_i is then 0 or 1. (We avoid recurring 1's in order to make the representation unique.) Put $t_i = 2b_i$ for each i and regard $0.t_1t_2t_3\ldots$ as the representation of a real number $f(x)$ to base 3. Then each t_i is 0 or 2. Now, since $t_1 \neq 1, f(x) \notin [\frac{1}{3},\frac{2}{3})$;† and since $t_2 \neq 1$, $f(x) \notin [\frac{1}{9},\frac{2}{9})$ or to $[\frac{7}{9},\frac{8}{9})$; and so on. Thus $f(x)$ does not belong to any of the deleted intervals, and so $f(x) \in F$. We have therefore defined a mapping $f: [0,1) \to F$. Under this mapping, distinct points of $[0,1)$ clearly go into distinct points of F; and hence there are as many points in F as in the whole of $[0,1)$

5. Turning a line segment

In 1917 the Japanese mathematician Kakeya proposed the following problem. Find the region with least area within which it is possible to turn a line segment of length 1 unit continuously through a complete revolution. Certainly the area of this region will be less than or equal to $\frac{1}{4}\pi$, which is the area of a disc of diameter 1 unit. It was quite soon shown to be, in fact, less than or equal to $\frac{1}{8}\pi$, which is the area inside a deltoid inscribed in a

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† $\notin$ means 'does not belong to'. Thus it is not true that $\frac{1}{3} \leqslant f(x) < \frac{2}{3}$.

circle of diameter $\frac{3}{2}$ units. Since, indeed, in Figure 3, the intercept AB

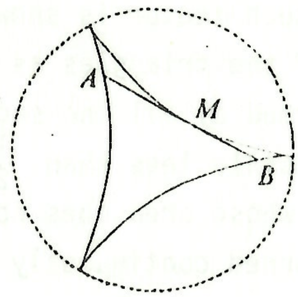

Figure 3

on the tangent at M is equal to 1 unit for each position of M, it is readily seen just how to rotate a unit line segment through 360° entirely inside the deltoid. But this was far from being the end of the story. The answer to Kakeya's question is that a unit line segment can be turned through 360° inside a region of arbitrarily small area. It will take a little time for us to justify this truly surprising assertion, for although the original argument has been considerably simplified over the years, yet it is by no means easy. It depends on a basic lemma, and we shall reserve the proof of this for an appendix (to be read by those of you who want more than a sketch of the argument).

Lemma. Let ABC be a triangle with base AB: divide AB into 2^n equal subintervals, and join each point of subdivision to C to form 2^n samll triangles inside ABC. For a suitable choice of n, it is possible to slide these small triangles along the base AB so that they overlap each other to such an extent that the total area which they cover in their new positions is arbitrarily small.

Now consider a circle with centre O and radius 1 unit. Describe an equilateral triangle ABC about this circle as shown in Figure 4 and join AO, BO, CO. Consider separately each of the triangles OBC, OCA, OAB with bases BC, CA, AB and, as in the lemma, divide each of them into 2^n small triangles by means of equally spaced points of subdivision of their bases. Figure 4 shows the subdivision of OBC into 2^3 small triangles. According to the lemma, given any positive number ϵ, however small, we may choose n so that these small triangles when translated parallel to the bases BC, CA, AB cover a total area not exceeding (say) $\frac{1}{2}\epsilon$. Our circle is divided by small triangles into sectors (one such sector is shown shaded in the figure), and we may consider translations of the triangles as translations of the sectors. Let us denote by U the figure formed by all the sectors in their final positions. Then the area of U is certainly less than $\frac{1}{2}\epsilon$. We are going to show how to enlarge U to a figure V whose area does not exceed ϵ and within which a unit line segment may be turned continuously through 360°. To this end,

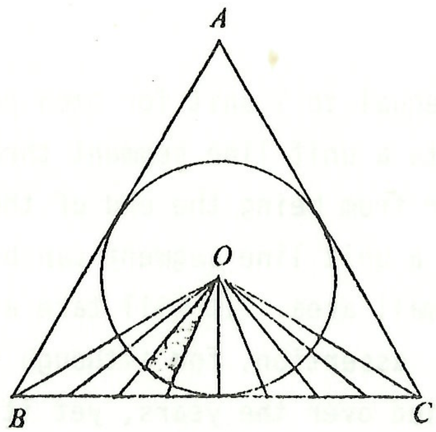

Figure 4

let us look first at two adjacent small sectors in their new positions. (it does not matter whether they correspond to small triangles in the same triangle, say OBC, in which case they will both have been translated in the same direction, or in neighbouring ones, in which case they will have been translated in different directions.) No directions will have been changed since the sectors have been translated only, not rotated. As in Figure 5, denote these two sectors by O'PQ, O"RS, where O'Q and O"R are parallel. Take two radii O'X, O"Y, one in each sector, which make a small angle θ. Let O'X, O"Y (perhaps produced) meet in O_1 and form a sector $O_1X_1Y_1$ at O_1 of radius 1 unit. Evidently O'P can be moved continuously to O"S entirely within the shaded region. To be specific: rotate O'P to O'X, slide O'X to O_1X_1, rotate O_1X_1 to O_1Y_1, slide O_1Y_1 to O"Y, and finally rotate O"Y to O"S. By considering all adjacent pairs of segments, let us augment U by means of all these small sectors $O_1X_1Y_1$ each with centre angle θ, to form V.

A unit line segment can evidently be turned through a complete revolution inside V. But the total area added to U by this procedure is equal to $3.2^n(\theta/2)$; so it only remains to choose θ to be less than or equal to $\epsilon/3.2^n$ to make this total area added to U to be less than or equal to $\frac{1}{2}\epsilon$. Thus V has area not exceeding ϵ, and we can indeed turn out line segment through a complete revolution inside a region of arbitrarily small area.

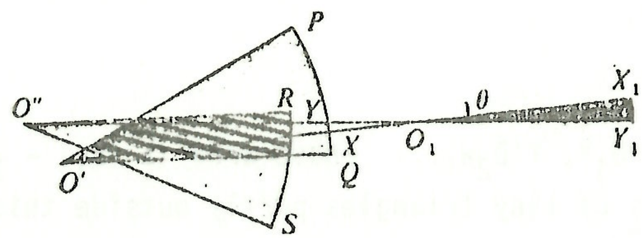

Figure 5

Appendix

We supply a proof of the lemma stated on p.74, omitting details of routine (albeit tedious!) elementary geometry. Let us denote by S the area of the triangle ABC and by c the length of AB. Now there is an even number of small triangles inside ABC, namely 2^n, and we begin by sliding the first, third, fifth (and so on) of these along AB through a small distance $cx/2^n$ towards B. To fix our attention, we shall look at the first four triangles to the right of A; so let us denote the first four points of subdivision (read from A) by X, Y, Z, W and indicate, as in Figure 6, the new positions of the triangles ACX and YCZ by corresponding dashed letters. Let $A'C'$ meet YC in D_1 and $Y'C'$ meet WC in D_2. Now the total area covered by $A'C'X'$ and XCY is equal to the area of the (shaded) triangle $A'D_1Y$ together with that of two tiny (darker shaded) triangles. We may calculate this first area to be $S(1 - \frac{1}{2}x)^2/2^{n-1}$, and the sum of the areas of the two tiny triangles to be $Sx^2/2^n$. Now, by a consideration of appropriate ratios, it is easily checked that YD_1 and $Y'D_2$ are equal in length; they are also parallel and so, by a further slide, the triangles $A'D_1Y$ and $Y'D_2W$ may clearly be 'fitted together'. This happens all the way along, and we form a new triangle from

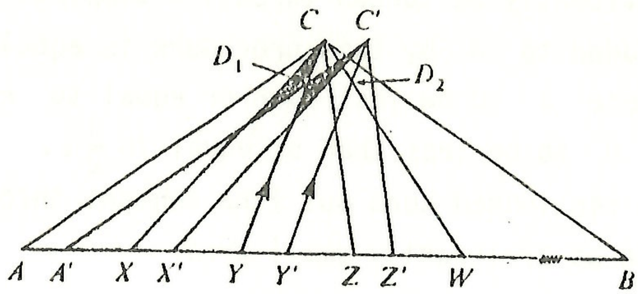

Figure 6

the 2^{n-1} triangles $A'D_1Y$, $Y'D_2W, \ldots$ whose area is $S(1 - \frac{1}{2}x)^2$, and in addition we have 2^{n-1} pairs of tiny triangles partly outside this triangle with total area at most $\frac{1}{2}Sx^2$. We repeat the whole procedure on the new triangle with its subdivision into 2^{n-1} small triangles, and go on repeating it. A moment's consideration will confirm that all the sliding involved is induced by sliding the original 2^n triangles ACX, XCY, YCZ,... in ABC. After $n - 1$ repetitions, the total area covered by these triangles in their new positions will be at most equal to

$$\underbrace{S(1 - \tfrac{1}{2}x)^2 \ldots (1 - \tfrac{1}{2}x)^2}_{n} + \tfrac{1}{2}Sx^2 + \tfrac{1}{2}S(1 - \tfrac{1}{2}x)^2x^2 + \ldots + \tfrac{1}{2}x)^{2n-2}x^2$$

$$< S(1 - \tfrac{1}{2}x)^{2n} + Sx^2/2(1 - (1 - \tfrac{1}{2}x)^2) .$$

Finally, given any positive number δ, however small, we may first choose x small enough to make the second term above less than $\delta/2$ and then choose n large enough to make the first term less than $\delta/2$. The whole area is then less than δ.

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