Parabola Volume 21, Issue 1 (1985)

THE WORLD OF MATHEMATICS

Here we are introducing the first of a series of short articles from the world of mathematics. These may deal with snippets of history, with development of technique, with the emergence of new concepts and/or anecdotes.

EUCLID OF ALEXANDRIA AND THE CONCEPT OF PROOF

BY

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Proof is an argument based on logic which establishes whether a given statement is true or false.

In order to proceed further with the concept of proof we first introduce the concepts of undefined terms and axioms.

In any mathematical system there are a number of undefined terms or elements. For example take the case of points and lines in geometry. We may define a point to be the the intersection of two lines. In Fig. 1 the two lines m and n define the point A. In turn we state that two points define a line. In Fig. 2 the two points P and Q define the line ℓ.

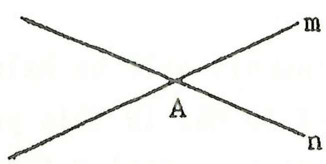

Fig. 1

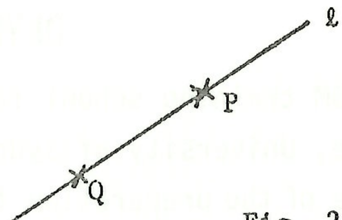

Fig. 2.

Observe that we have defined a point in terms of lines and in turn we have defined a line in terms of points. This is logically wrong since we have gone in a circle. To avoid going in circles we leave the terms of point and line as undefined. Better still if points are taken as undefined, lines may be defined in terms ofthem; alternatively, if lines are taken as undefined, points may be defined in terms of them.

Another way of presenting our argument for undefined terms is as follows:

To define a given term we use simpler terms and in turn we define these simpler terms by simpler terms and so on. There must be a stage where we put a stop to this endless process and leave some of the terms undefined. We just cannot go on and define everything. Each logical argument must have a starting point where we accept a number of undefined terms.

We now introduce the concept of an axiom. Sentences or statements expressing certain properties among the different terms are called propositions. A proposition may be true or false. For example two propositions in the number system are:

8 is an even number.

17 is not an odd number.

The first proposition is true while the second is false.

Some of the propositions were regarded in ancient times as self-evident truths and were accepted without proof. Such propositions were called axioms or postulates. For example in geometry the proposition:

"One straight line only can be drawn between two distinct points." is an axiom.

Consider the points P and Q. There are infinitely many possible paths joining P and Q. c_1, which is a straight line, c_2 and c_3 are three of these paths as shown in Fig.3.

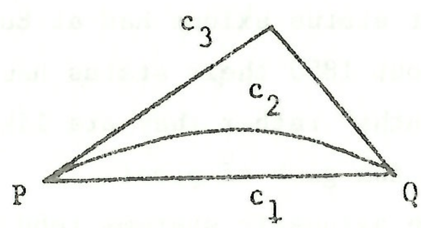

Fig. 3

By intuition we know that c_1 is the shortest path joining P and Q. So we may, as Archimedes did, formulate another axiom in geometry as follows;

"Of all the paths joining two distinct points, the straight line path is the shortest."

The axioms are a foundation from which other propositions often called theorems are derived by logical arguments.

Once a theorem is proved it may be used together with other axioms and previously-proved theorems to prove new theorems.

This type of logical argument is called deductive reasoning. Given a logical system, a set of undefined elements and a set of axioms are first chosen. then they are used to derive, by logical reasoning, theorems in that system.,

Euclid attempted to introduce such a system of deductive reasoning in geometry. This was done in his treatise consisting of thirteen books or chapters called the Elements written about 23 centuries ago. Over these centuries there have been several hundreds of editions of Euclid's Elements. One of these editions (in three volumes) is: The Thirteen Books of Euclid's Elements by Thomas L. Heath (Dover, New York, 1956).

There was a precursor to Euclid. Hippocrates of Chios (about 430 B.C.) apparently wrote a text similar in style to the Elements. Unfortunately, it was lost quite early on in Greek history.

Euclid did recognise the necessity for axioms in his deductive system but unfortunately he did not fully realise the concept of undefined terms. Instead of undefined terms he starts the first book with a set of definitions (often not clear; for example he defines a point as that which has no magnitude) then a set of ten axioms (the first five are called postulates and the last five are called common notions). From there onwards he proceeds to derive all his theorems from the axioms and definitions.

Euclid's third postulates may be stated as follows:

"A circle may be drawn with any point as centre and with any radius."

Euclid's fifth common notion is:

"The whole is greater than the part".

Postulates are peculiar to geometry while common notions are common to 'all' mathematics.

It is not clear just what status axioms had at Euclid's time, it is certainly the case that since about 1850 their status has changed. No longer need they be "self-evident truths" rather they are like undefined terms, they are the starting positions in the game of proof.

Of course randomly chosen axiomatic systems tend not to lead to deep or applicable mathematics. If a formal logical system is to be a model of some 'reality' then, a set of undefined elements and a set of axioms are first chosen. There are infinitely many ways of choosing such sets. However, a lot of insight, observations and experimentation is required to arrive at a suitable set containing a few elements. Euclid was able to achieve this feat in geometry. The synthesising and organizational genius of Euclid in taking the disparate theorems of geometry, some of which were already 300 years old, and structuring them into a logically cohesive package based on 10 simple axioms is truly an intellectual achievement of the first rank.

We now discuss an example of proof. For this purpose we choose the first part of Proposition 5, Book 1 of Euclid which states: "The angles at the base of an isosceles triangle are equal." A model of the proof is as follows:

Given or Hypothesis: Triangle ABC with AC = BC as shown in Fig.4.

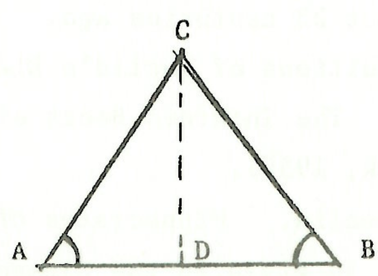

Fig. 4

To Prove: That $\angle CAB = \angle CBA$

PROOF: We draw CD bisecting $\angle ACB$ and meeting AB in D. In the triangles ADC and BDC we have:

$$AC = BC \quad \text{(Hypothesis)}$$

$$DC = DC \quad \text{(Common}$$

$$\angle ACD = \angle BCD \quad \text{(Construction)}$$

$\therefore$ Triangles ADC and BDC are equal and so $\angle CAB = \angle CBA$ (By Proposition 4. Book I of Euclid).

Q.E.D.

We mention that Euclid's proof was more complicated than the above proof, In spite of Euclid being a model of rigor and certitude the conclusion does not follow logic from axiom.

In the above proof there are hidden axioms. For example it is assumed that the two lines AB and the bisector of $\angle ACB$ intersect. This is because of the axiom of continuity. We explain as follows:

Take the case of a circle and two points A and B. The point A being inside the circle and B outside it as shown in Fig.5.

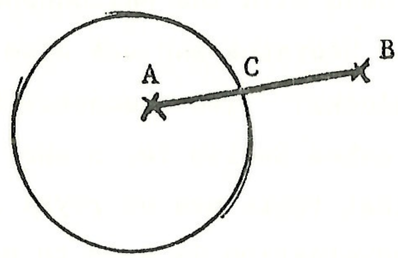

Fig. 5

If we join AB then this line must intersect the circle. This is because of the axiom of continuity.

Euclid assumed the axiom of continuity right from Proposition 1, Book 5, where he describes the construction of an equilateral triangle. In his proof he assumes that two circles will intersect.

Was Euclid aware of logical gaps in his proof? Did he realise he was making hidden assumptions and did he for sake of clarity and ease of exposition ignore them? Or did he fail to see these axioms among which is the axiom of continuity.

Unfortunately little is known about the life of Euclid. Most of what we know about this Greek mathematician is due to the Greek philosopher Proclus (412-485 A.D.) and a little due to the Greek mathematician Pappus (about 300 A.D.).

We do not know his birthplace. We are also uncertain about the dates of his birth and death. The dates usually given are 330-270 B.C. He came after Plato (c. 428-348/347 B.C.) and before Archimedes (287-217 B.C.). He learned geometry (probably in Athens) from the pupils of Plato.

After the death of Alexander the great (356-323 B.C.) Ptolemy I became the king of Egypt. He established the famous museum and library at Alexandria where Euclid became a teacher of mathematics. There he proved to be a very successful teacher. His greatest work is the Elements. This textbook has remained almost unchanged for about 2300 years and as mentioned earlier several hundreds of editions of this textbook have been printed. The above institution was partly destroyed when Caesar besieged Alexandria in the year 47 B.C.

The Elements does not include all the geometry known in Euclid's time. For example it does not include conics. Most of the work in the Elements is not original. For example he used many results obtained earlier by Eudoxus (408-355 B.C.). It is rather an attempt by Euclid to introduce a logical system in geometry based on deductive reasoning as explained earlier.

Our Euclid is often confused with the philosopher Euclid of Megara (about 400 B.C.) who was a student of Socrates and was also present at Socrates' death (when he drank the poison hemlock). Two anecdotes about our Euclid are:

i) King Ptolemy I once asked Euclid for a shorter way in geometry, Euclid's reply was that there was no royal road to geometry. However a similar conversation is said to have taken place between Alexander the Great and Menaechmus.

ii) One of his pupils, after learning the first theorem in Euclid's Elements asked him what he will get for learning it. Euclid told his slave to give the pupil a coin since he must make gain by what he learns.

Euclid gave the world a prototype of a logical system which is still used in proving theorems.

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