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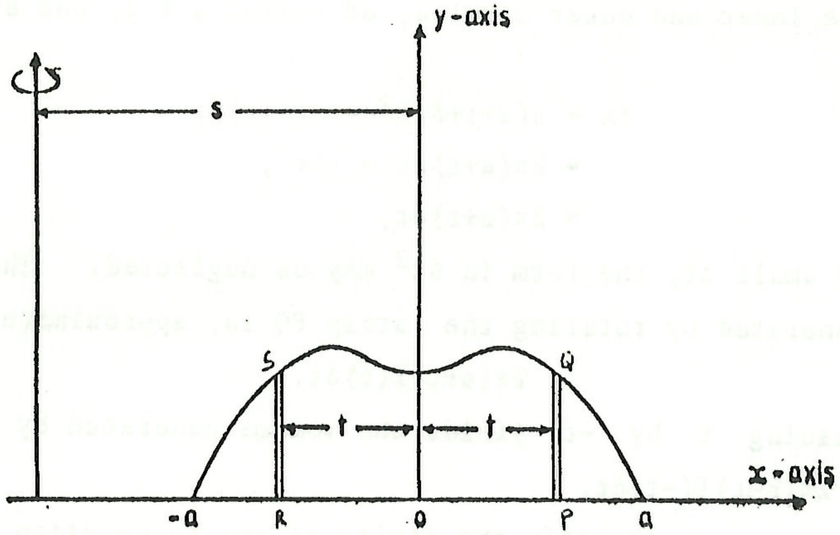
This issue considers some problems concerning applications of mathematics.

<u>Question 85.1.</u>



The diagram shows the area A between the smooth curve $y = f(x)$, $-a \leqslant x \leqslant a$, and the x-axis. (Note that $f(x) \geqslant 0$ for $-a \leqslant x \leqslant a$ and $f(-a) = f(a) = 0$.) The area A is rotated about the line $x = -s$ (where $s > a$) to generate the volume V. This volume is to be found by slicing A into thin vertical strips, rotating these to obtain cylindrical shells, and adding the shells. Two typical strips of width δt whose centre lines are distance t from the y-axis are shown.

(a) Show that the indicated strips generate shells of approximate volume $2\pi f(-t)(s-t)\delta t$, $2\pi f(t)(s+t)\delta t$, respectively.

(b) Assuming that the graph of f is symmetrical about the y-axis, show that $V = 2\pi sA$.

Problems of this type are generally fairly simple mathematically, but do require a thorough understanding of the applications of calculus to the real world. In this case, consider the strip PQ, and its rotation about the line $s = -a$.

(a) Rotation of the strip PQ generates a thin cylinder, of thickness δt and height $f(t)$. The cylinder is a 'right cylinder', i.e. its volume is the product of its height and the base area. Thus the volume of the thin cylinder of rotation is $f(t)\delta A$ where δA is the area of the crossection., which in this case is a thin annulus. The area δA of this annulus is given by the difference of the areas of the inner and outer circles, of radius $s + t$, and $s + t + \delta t$, respectively, as

$$\begin{aligned}\delta A &= \pi(s+t+\delta t)^2 - \pi(s+t)^2,\\ &= 2\pi(s+t)\delta t + \pi\delta t^2,\\ &\approx 2\pi(s+t)\delta t,\end{aligned}$$

since, for very small δt, the term in δt^2 may be neglected. Thus the volume of the cylinder generated by rotating the strip PQ is, approximately,

$$2\pi(s+t)f(t)\delta t.$$

Similarly, replacing t by $-t$ yields the volume generated by rotating the strip QR to be $2\pi(s-t)f(-t)\delta t$.

The next step is to identify the limits of the integration which produces the required volume.

Note that $V = \int_{-a}^{a} 2\pi(s+t)f(t)dt$, and, <u>also</u>, $V = \int_{-a}^{a} 2\pi(s-t)f(-t)dt$.

Thus

$$2\pi\int_{-a}^{a} [(s+t)f(t) + (s-t)f(-t)]dt = 2V!$$

Thus

$$V = \pi\int_{-a}^{a} [(s+t)f(t) + (s-t)f(-t)]dt \qquad (1)$$

Alternatively (justify this to yourself)

$$\begin{aligned}V &= 2\pi\int_{0}^{a} (s+t)f(t)dt + 2\pi\int_{0}^{a} (s-t)f(-t)dt\\ &= 2\pi\int_{0}^{a} [(s+t)f(t) + (s-t)f(-t)]dt \qquad (2)\end{aligned}$$

The expressions in (1) and (2) for V are equal. Can you justify this? [If not put $t = -u$ in (2)].

(b) The graph of f is symmetric about the y axis, so that $f(t) = f(-t)$. Thus, from (1),

$$V = 2\pi s\int_{-a}^{a} f(t)dt = \underline{2\pi sA}.$$

Question 85.2.

This is a hardish problem on projectile motions.

Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection (upward inclination to the horizontal), α, but with different velocities U, V metres per second $(U < V)$.

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

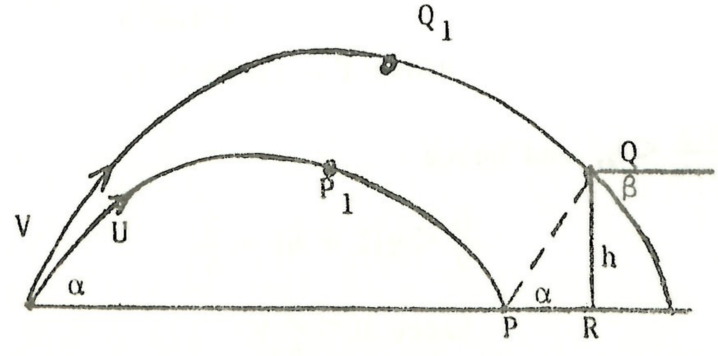
(a) Show that, while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time. Hence, express the horizontal distance from P to the foot of the wall in terms of h, α.

(b) Show that

$$V(\tan\alpha + \tan\beta) = 2U\tan\alpha,$$

and deduce that, if $\beta = \frac{1}{2}\alpha$, then

$$U < \frac{3}{4}V.$$



Let (x,y) be the coords of P_1, (p,q) be the coords of Q_1 at time t, then

(a)

$$x = U \cos \alpha \; t$$
$$y = U \sin \alpha \; t - \tfrac{1}{2}gt^2$$
$$p = V \cos \alpha \; t$$
$$q = U \sin \alpha \; t - \tfrac{1}{2}gt^2$$

Thus, the gradient of P_1Q_1 is $\frac{q-y}{p-x} = \frac{t(V-u)\sin \alpha}{t(V-u)\cos \alpha} = \tan \alpha.$

Hence P_1Q_1 always makes an angle α with the horizontal. Thus $\underline{PR = h \cot \alpha}$ is the required distance.

(b) This section is straight forward, really, but quite hard, because it requires a change of thought patterns! At time t, the velocity components (u,v) of the upper particle are given by

$$u = V \cos \alpha$$
$$v = V \sin \alpha - gt$$

and thus

$$\tan \beta = \frac{-v}{u} = \frac{-V \sin \alpha + gT}{V \cos \alpha}$$

at Q, when T is the time of flight of the lower particle. Remembering the formula for T, or letting $y = 0$ in (a) above, we find that $T = 2U \sin \alpha / g$

$$V \tan \beta = -V \tan \alpha + 2U \tan \alpha, \text{ and hence}$$
$$\underline{V(\tan \alpha + \tan \beta) = 2U \tan \alpha}$$

Thus
$$\frac{U}{V} = \frac{\tan \alpha + \tan \beta}{2 \tan \alpha} = \tfrac{1}{2}\left(1 + \frac{\tan \beta}{\tan \alpha}\right) .$$

Given that $\beta = \tfrac{1}{2}\alpha$, then $\tan \alpha = \tan 2\beta = \frac{2 \tan \beta}{1-\tan^2\beta} > 2 \tan \beta$

since $1 - \tan^2\beta < 1.$

Thus $\frac{\tan \beta}{\tan \alpha} < \tfrac{1}{2}$, and hence

$$\frac{U}{V} < \tfrac{1}{2}(1 + \tfrac{1}{2}) = \frac{3}{4}$$

$$\text{hence } \underline{U < \frac{3}{4} V}$$