PROBLEM SECTION

Solutions of one or more of the following problems are invited. Answers will appear in Vol. 21 No. 3.

Q. 624. Three motorists A, B, and C often travel on a certain highway, and each motorist always travels at a constant speed. A is the fastest of the three, and C the slowest.

One day B overtakes C, five minutes later A overtakes C and in another 3 minutes A overtakes B. The next day, A overtakes B first, then, nine minutes later, overtakes C. When will B overtake C ?

Q. 625. At a party, each boy shakes hands with an odd number of girls, and each girl shakes hands with an odd number of boys. Show that the total number of boys and girls is even.

Q. 626. Five coins appear to be identical, but two of them are counterfeits. One is lighter and one is heavier than a good coin, but together they exactly counterbalance two good coins. Show how to identify all the coins in three weighings, using a beam balance.

Q. 627. The probability that any letter reaches its destination is 4/5. I post a letter to a friend. If he received it he would certainly have sent a reply, but I receive no reply. What is the probability that he received my letter?

Q. 628. Prove that if (x_1,y_1); (x_2,y_2); and (x_3,y_3) are three points in the Cartesian plane which are vertices of an equilateral triangle, it is impossible that all the co-ordinates are integers.

Q. 629. A "uni-digit number" is one whose ordinary decimal expression consists of only one digit repeated a number of times; e.g., 44 or 7777 or 9,999,999. Prove that a uni-digit number greater than 10 is not a perfect square.

Q. 630. n! (read n factorial) is defined to be the product of all the integers from 1 to n inclusive; i.e.,

$$n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n.$$

(a) Which is larger, $\sqrt[8]{(8!)}$ or $\sqrt[9]{(9!)}$?

(b) Simplify $1(1!) + 2(2!) + 3(3!) + \ldots + n(n!)$.

Q. 631. Show how to construct a triangle given one angle, the length of the opposite side, and the difference of the lengths of the other two sides.

Q. 632. Prove that $\sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} = 2$.

Q. 633. (a) Prove that there exist two powers of 3 having the same first 5 digits.

(b) Show that there exists a power of 3 whose first 4 digits are 1111.

Q. 634. No three of the $n(n-3)/2$ diagonals of a convex n-gon are concurrent at a point inside the figure. Find

(a) the number of points of intersection of diagonals inside the n-gon.

(b) the number of compartments into which the interior of the n-gon is dissected by the diagonals.

(For example, when $n=4$ the answers are 1 and 4 ;
when $n=5$ " " " 5 and 11.)

Q. 635. (a) Show that $5^n - 1$ is divisible by 4, n being any positive integer.

(b) A list of prime numbers $p_1, p_2, p_3, \ldots, p_n, \ldots$ is generated as follows:
$p_1 = 2$, and if $n > 1$, p_n is the largest prime factor of $p_1 p_2 \ldots p_{n-1} + 1$. Thus $p_2 = 3$, (the largest prime factor of $2+1$), $p_3 = 7$, $p_4 = 43$, $p_5 = 139$, (since $2.3.7.43 + 1 = 1807 = 13.139$.)
It does not follow from this rule that p_{n+1} is larger than p_n. However, prove that the prime number 5 never occurs in the list.