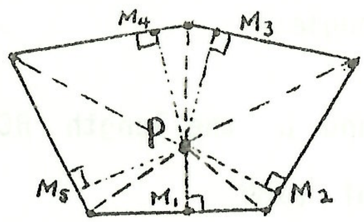
SOLUTIONS TO PROBLEMS FROM VOLUME 20, NUMBER 2

Q.600. P is a point inside a convex polygon all of whose sides are of equal length. Perpendiculars are constructed from P to the sides of the polygon (produced if necessary). Show that the sum of the lengths of the perpendiculars ($PM_1 + PM_2 + \ldots + PM_5$ in the figure) is the same for all positions of P.



Solution: Join P to each of the vertices of the polygon, thus dissecting it into triangles. If each of the sides has length ℓ, the area of the polygon is given by the sum of the areas of the triangles $\frac{1}{2}\ell \times (PM_1 + PM_2 + \ldots + PM_n)$. Hence for any position of P inside the polygon the sum of the lengths of the perpendiculars to the sides is $2 \times$ Area of polygon $\div \ell$.

Correct solutions received from: C. Playoust (Loreto Convent, Kirribilli); D. Stephens (Knox Grammar School).

Q.601. Using your calculator you will be able to check that

$$\sqrt{5+\sqrt{21}} + \sqrt{8+\sqrt{55}} \quad \text{and} \quad \sqrt{7+\sqrt{33}} + \sqrt{6+\sqrt{35}}$$

are approximately equal. Either prove that they are exactly equal, or decide (with proof) which is the larger.

Solution: Note that $(\sqrt{a}+\sqrt{b})^2 = a + b + 2\sqrt{ab}$ and hence that

$$\sqrt{\frac{a+b}{2} + \sqrt{ab}} = \frac{(\sqrt{a}+\sqrt{b})}{\sqrt{2}}.$$

Using this $\sqrt{5+\sqrt{21}} = \sqrt{\frac{3+7}{2} + \sqrt{3\times 7}} = \frac{\sqrt{3}+\sqrt{7}}{\sqrt{2}}$,

and similarly $\sqrt{8+\sqrt{55}} = \frac{(\sqrt{5}+\sqrt{11})}{\sqrt{2}}$;

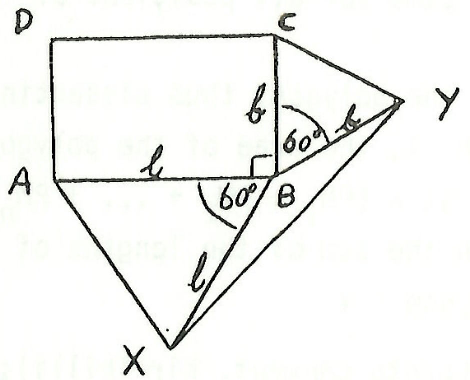
$$\sqrt{7+\sqrt{33}} = \frac{\sqrt{3}+\sqrt{11}}{\sqrt{2}};$$

and $\sqrt{6+\sqrt{35}} = \dfrac{\sqrt{5}+\sqrt{7}}{\sqrt{2}}$.

It follows that both of the given expressions are equal to $\frac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{5}+\sqrt{7}+\sqrt{11})$.

Q.602. ABCD is any rectangle, and ABX, BCY, CDV, and DAW are outward drawn equilateral triangles. Prove that the sum of the areas of the triangle AXW, BYX, CVY and DWV equals the area of the rectangle.

Solution: Let ℓ represent the length AB, and b the length BC.



Then area of ΔXBY

$= \frac{1}{2}lb\sin X\hat{B}Y$

$= \frac{1}{2}lb \sin(360° - 90° - 2\times 60°)$

$= \frac{1}{2}lb \sin 150° = \frac{1}{4}lb$.

Since all four triangles are obviously congruent their areas total lb, the area of the rectangle.

Correct solutions from A. Chow (Sydney Grammar School), P. Nathanial (Crow's Nest Boys' High School), C. Playoust (Loreto Convent), D. Stephens (Knox Grammar School).

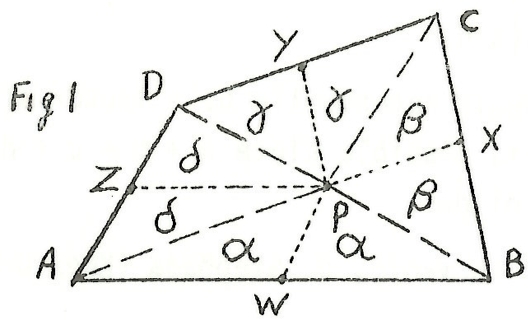
Q.603. Show that there is no party of 10 members in which the members have 9, 9, 9, 8, 8, 8, 7, 6, 4, 4, acquaintances among themselves. (Acquaintances may be assumed to be mutual).

Solution: Let A, B, C, D, E, F, G, H, I, J have respectively 9, 9, 9, 8, 8, 8, 7, 6, 4, 4 acquaintances. Then A is acquainted with all 9 other people at the party. Thus both I and J are acquainted with A, and similarly with B and with C. Next D having 8 acquaintances must know at least one of I and J. Similarly for E and F. Hence together I and J's acquaintances total at least 2 + 2 + 2 + 1 + 1 + 1 = 9. Since the given numbers 4 and 4 total less than 9, the list of figures given is impossible.

Correct solutions from L-A. Koe (James Ruse Agricultural High School), C. Playoust (Loreto Convent), D. Stephens (Knox Grammar School).

Q.604. Given a convex quadrilateral ABCD show how to construct a point P in its interior such that the lines joining P to the mid points of the sides divide the quadrilateral into 4 equal areas.

Solution: Let P be the required point. Join P to each of the vertices as well as to the midpoints W, X, Y, Z of the sides of the quadrilateral. (see figure) Let the area of each of the triangles into which the figure is decomposed be represented by the Greek letter contained in the diagram. Since it is easy to see that Area ΔPAW = Area ΔPBW, they both contain the same Greek letter, α. Similarly for the two β's, γ's, and δ's.

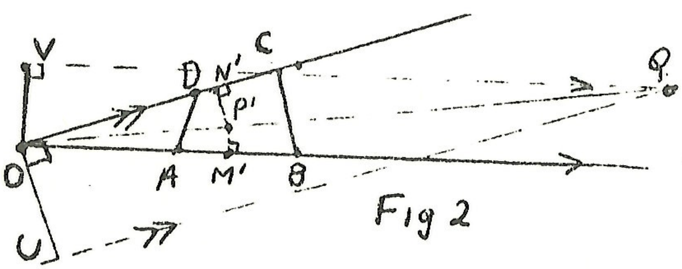


From data, $\alpha + \delta = \alpha + \beta = \beta + \gamma = \gamma + \delta$. We see immediately that these conditions are equivalent to $\alpha = \gamma$ and $\beta = \delta$. Hence we must construct P so that

(1) Area ΔPCD = Area ΔPAB (i.e. $2\gamma = 2\alpha$)

and (2) Area ΔPAD = Area ΔPBC ($2\delta = 2\beta$).

To achieve (1), refer to Figure 2.



Produce CD and BA to meet at O. Construct $OV \perp OA$ with $OV = CD$, and $OU \perp OC$ with $OU = AB$. Construct lines VQ and UQ parallel to OA, OD respectively, meeting at Q. Join OQ. Then if P' is any point on OQ,

Area $\Delta P'CD$ = Area $\Delta P'AB$

since the lengths of the perpendiculars P'M', and P'N' are in inverse proportion to the bases AB and CD, by our construction. (Check this).

Similarly construct the straight line locus of P" such that Area $\Delta P''AD$ = Area $\Delta P''BC$. The intersection of the two straight lines determines the position of P. (Some modification of the construction is required if either pair of opposite sides of ABCD are parallel lines. Can you supply it?)

Q.605. If a, b, c, d are four positive integers such that $ab - cd$, prove that $a + b + c + d$ is not a prime number.

Solution. Note that

$$a(a+b+c+d) = a^2 + ab + ac + ad = a^2 + cd + ac + ad$$
$$= (a+c)(a+d)$$

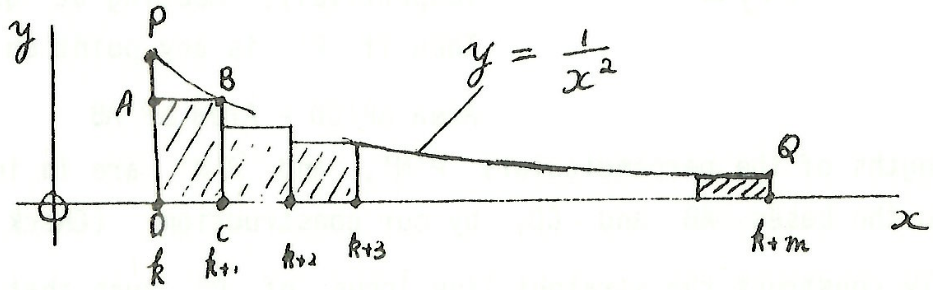
If $a + b + c + d$ were a prime number, it would necessarily be a factor of either $(a+c)$ or $(a+d)$, neither of which is possible since $a + b + c + d$ is larger than either of them. The result follows.

Correct solutions from: C. Playoust (Loreto Convent), D. Stephens (Knox Grammar School).

Q.606. Suppose we start adding $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$ where the n th term is $1/n^2$. The more terms added, the closer the total gets to $\pi^2/6$. (This requires advanced methods to prove; the result was obtained by L. Euler). Show that if only k terms are added, the total differs from the eventual limit sum by less than $1/k$, where k is any positive integer. i.e. show that

$$\frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \frac{1}{(k+3)^2} + \ldots < 1/k,$$ however many terms on the left hand side are added.

Solution. EITHER (Using Calculus)



The figure contains the graph of $y = \frac{1}{x^2}$ for $k \le x \le k+m$. Note that $\frac{1}{(k+1)^2}$ is the area of the rectangle ABCD, since its height BC is $\frac{1}{(k+1)^2}$

and its length DC is 1.

In fact $\frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \ldots + \frac{1}{(k+m)^2}$ is the sum of the areas of the shaded rectangles, and it is clear that this is less than the area under the graph of $\frac{1}{x^2}$ between $x = k$ and $x = k + m$.

Hence $$\frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \ldots + \frac{1}{(k+m)^2} < \int_k^{k+m} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_k^{k+m}$$

$$= \frac{1}{k} - \frac{1}{k+m} < \frac{1}{k}$$

Q.E.D.

OR (A purely algebraic argument)

$$\frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \ldots + \frac{1}{(k+m)^2} < \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \ldots + \frac{1}{(k+m-1)(k+m)}$$

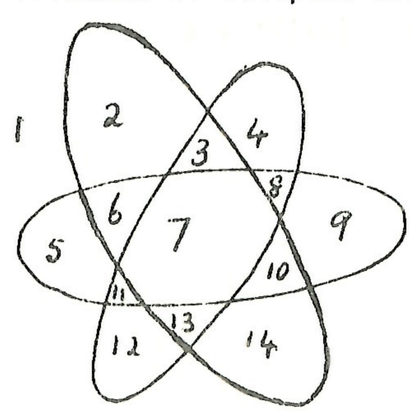
(since every term has been increased by diminishing the denominator)

$$< \left(\frac{1}{k} - \frac{1}{k+1}\right) + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \ldots + \left(\frac{1}{k+m-1} - \frac{1}{k+m}\right)$$

$$< \frac{1}{k} - \frac{1}{k+m} < \frac{1}{k}.$$

Q.E.D.

Q.607. A number of ellipses are drawn in the plane, any two of them intersecting in 4 points. No three of the curves are concurrent. In the figure, three such ellipses have been drawn, and the plane has been divided by them into 14 regions (including the unbounded region lying outside all the ellipses). Into how many regions would the plane be divided if 10 ellipses were drawn.



Solution: Let R_n be the number of regions resulting from n ellipses. For example, $R_1 = 2$, $R_2 = 6$, and $R_3 = 14$.

The n th ellipse intersects each of the previously drawn ellipses in 4 distinct points, and these $4(n-1)$ points on it divide its perimeter into $4(n-1)$ arcs. Each of these arcs cuts across one of the regions resulting from the first $(n-1)$ ellipses, dividing it into 2 pieces, and thus altogether $4 \times (n-1)$ new region are created by the n th ellipse.

Hence $R_n = R_{n-1} + 4(n-1)$ for all n, and repeated use of this recurrence relation yields

$$R_n = 4(n-1) + 4(n-2) + R_{n-2} = \ldots = 4(n-1) + 4(n-2) + 4(n-3) + \ldots + 4 \times 1 + R_1$$

$$= 4(\overline{n-1} + \overline{n-2} + \ldots + 2+1) + 2.$$

$$= 4.(n-1) \times \left(\frac{1+\overline{n-1}}{2}\right) + 2$$

$$= 2n^2 - 2n + 2 \quad \text{for all } n.$$

In particular $R_{10} = 182$.

Correct solutions received from: L-A. Koe (James Ruse Agricultural High School), C. Playoust (Loreto Convent).

<u>Q.608</u>. Find all integers x such that

$$x^4 + 4x^3 + 6x^2 + 4x + 5$$

is a prime number.

<u>Solution</u>: Note that $x^4 + 4x^3 + 6x^2 + 4x + 1 + 4 = (x+1)^4 + 4$

$$= [(x-1)^4 + 4(x+1)^2 + 4] - 4(x+1)^2$$

$$= [(x+1)^2 + 2]^2 - [2(x+1)]^2$$

$$= [(x+1)^2 - 2(x+1) + 2][(x+1)^2 + 2(x+1) + 2]$$

$$= (x^2+1)(x^2+4x+5)$$

When x is an integer, both factors are positive integers, and their product has no chance of being prime unless one of the factors is 1.

$x^2 + 1 = 1 \Rightarrow x = 0$ and the value is then the prime number 5.

$x^2 + 4x + 5 = 1 \Rightarrow x = -2$ again yielding the prime value 5.

There are no other values of x yielding a prime number for the value of the expression.

Q.609. Let x denote the number 11111111 $\left(= \frac{10^8-1}{9} \right)$. If y is any multiple of x, show that the sum of the digits of y is not equal to 63. Find a multiple of x the sum of whose digits is 65.

Solution: If y is any positive integer expressed in the usual decimal notation we denote by $S(y)$ the sum of its digits. We shall prove that :-

Lemma If y is a multiple of x with more then 8 digits there exists a smaller positive multiple of x, z say, such that $S(z) = S(y) - 9 \times k$ for some non-negative integer k.

[Proof: Let $y = a_n \times 10^n + a_{n-1} \times 10^{n-1} + \ldots + a_1 \times 10 + a_0$ where $a_n, a_{n-1}, \ldots, a_1, a_0$ are the decimal digits of y, $a_n > 0$, and $n \geq 8$.

Let $z = y - 9x \times 10^{n-8} = y - 10^n + 10^{n-8}$. Clearly z is a multiple of x.

Let $z = b_n \times 10^n + b_{n-1} \times 10^{n-1} + \ldots + b_{n-8}10^{n-8} + \ldots + b_0$

Case I $a_{n-8} < 9$.

In this case $b_n = a_n - 1$, $b_{n-8} = a_{n-8} + 1$ and the other digits of z are equal to the corresponding digits of y. So $S(z) = S(y) = S(y) - 9 \times 0$.

Case II There are k (> 0) consecutive digits in y, the last of which is a_{n-8}, all equal to 9. Then adding 10^{n-8} to y replaces all these 9's by 0's and a "carried" 1 is added to the next digit of $y - 10^n$, $a_{n'}$. Hence b_n is one less than a_n, some digit $b_{n'}$ is one greater than $a_{n'}$, and k q's in y have been replaced by k o's in z. Thus $S(z) = S(y) - 9 \times k$.

(Slight rewording is needed if $k = 8$ in this)]

Armed with this lemma, we are ready to tackle both parts of the question.

First Part. Let $S(y) = 63$. We continue to apply the lemma until eventually some positive number z' is obtained of only 8 digits, which is a multiple of x and such that $S(z') = 63 - 9k_1 - 9k_2 \ldots$. Note that $S(z')$ is a multiple of 9 not greater than 63. But the only 8 digit multiple of x for which the sum of digits is a multiple of 9 is 99999999, the sum of digits being 72 (obviously greater than 63). Hence no such y can exist.

Second Part: If $S(y) = 65$, the eight digit number z' obtained by repeated application of the lemma has $S(z') = 65 - 9k'$ (which is 2 more than a multiple of 9), and also $S(z')$ is a multiple of 8, since all 8 digits of z' are the same. The only possibility is $z' = 77777777$. Adding $8 \times (10^8 - 1)$ to this gives $y = 877777769$, which has the required properties.

Q.610. Find all triples of positive integers (x, y, z) such that $3^x + 4^y = 5^z$.

Solution: For the sake of brevity, I will leave it to the reader to supply proofs of the first few assertions in what follows. For any z, 5^z exceeds a multiple of 4 by 1, but $3^x + 4^y$ exceeds a multiple of 4 by 1 if and only if x is even. Hence we let $x = 2X$. For any y, $3^x + 4^y$ exceeds a multiple of 3 by 1, but 5^z does so if and only if z is even. Hence we let $z = 2Z$.

Now $3^x + 4^y = 5^z \Rightarrow 2^{2y} = (5^Z)^2 - (3^X)^2 = (5^Z - 3^X)(5^Z + 3^X)$. Thus both $5^Z - 3^X$ and $5^Z + 3^X$ must be powers of 2, say

$$5^Z - 3^X = 2^u$$

$$5^Z + 3^X = 2^v$$ where u, v are positive integers, $v > u$, $u + v = 2y$.

Subtracting these, we obtain easily $3^X = 2^{u-1}(2^{v-u} - 1)$, whence, since the LHS is odd, $u = 1$. It follows that $v = 2y - 1$ and that

$$3^X = (2^{y-1})^2 - 1 = (2^{y-1} - 1)(2^{y-1} + 1).$$

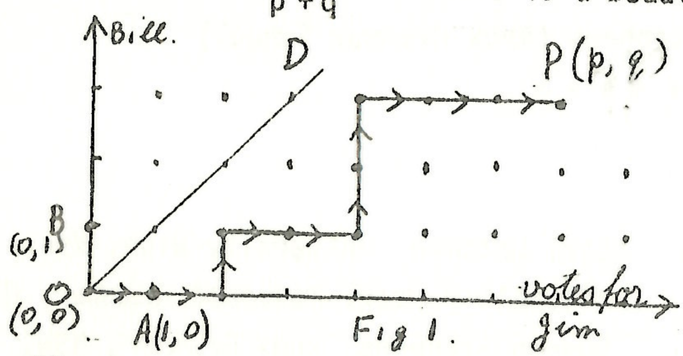
Both factors on the R.H.S. of this must be powers of 3, and since they differ by only 2 they must be $3^0 = 1$ and 3^1. Hence we must have $y = 2$, $X = 1$, after which it readily follows that the only solution in positive integers (x, y, z) of $3^x + 4^y = 5^z$ is the familiar one: $(x, y, z) = (2, 2, 2)$.

Q.611. Jim wins a secret ballot election with 7 votes, his only opponent Bill getting only 3 votes. Find the probability that as the votes were counted, Jim's tally remained greater than Bill's right from the first vote.

Solution. A straightforward calculation can be used. Jim's tally remains greater than Bill's if the first few votes counted are J J J J ..., or J J J B J..., or J J J B B J ..., or J J B J J ..., or J J B J B J ...; the probabilities of these events being $\frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7}$, $\frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6}$, $\frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{3}{9} \times \frac{2}{6} \times \frac{4}{5}$, $\frac{7}{10} \times \frac{6}{9} \times \frac{3}{8} \times \frac{5}{7} \times \frac{4}{6}$, and $\frac{7}{10} \times \frac{6}{9} \times \frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} \times \frac{4}{5}$ respectively. Adding these probabilities gives the answer $\frac{4}{10} = \frac{7-3}{7+3}$.

This result gneralizes as follows:

If p votes favour Jim, and q votes favour Bill with $p > q$, the probability that in the count Jim's tally is always greater than Bill's is given by the formula $\frac{p-q}{p+q}$. There is a beautiful argument which establishes this.



Each possible sequence of counting the $p + q$ votes can be represented by a unique broken line path on the integer lattice from the point (0,0) to the point (p,q). (See Figure with $p = 7$, $q = 3$). For example, the path shown on the figure corresponds to a count J J B J J B B J J J. The probability of this path is

$$\frac{7}{10} \times \frac{6}{9} \times \frac{3}{8} \times \frac{5}{7} \times \frac{4}{6} \times \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} \times \frac{2}{2} \times \frac{1}{1} = \frac{7!\,3!}{10!},$$

and one sees that exactly the same answer is obtained for the probability of any other path. Hence the probability that Jim's tally is always ahead is given by $\frac{N}{T}$ where T is the total number of paths from O to P and N is the number which after leaving O remain strictly to the right of the diagonal line OD (henceforth referred to as "good paths").

First $T = {}^{p+q}C_q = \frac{(p+q)!}{p!q!}$; since it is the number of sequences of pJ's and qB's; there are ${}^{p+q}C_q$ ways of choosing in which q positions to place the B's.

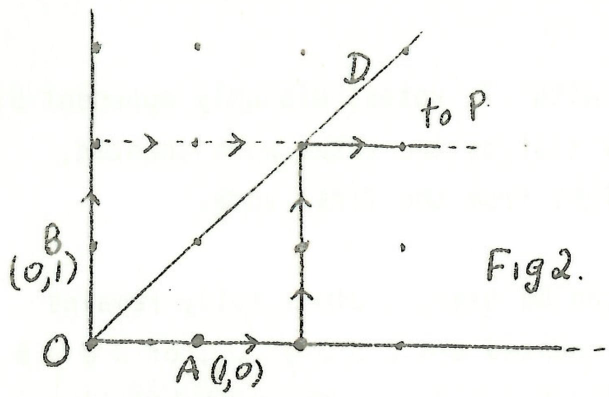


Fig 2.

Now to find N. All the good paths certainly pass through A(1,0). There are altogether $^{p+q-1}C_q$ paths from A to P but some of them are "bad" in that they meet the diagonal line OD. Let Q be the first point on the diagonal on some bad path from A (See figure 2). If one reflects the portion A - Q of this path in the diagonal OD one obtains a path from B to Q (and then to P). Conversely every path from B to P must cross the diagonal at some point, and reflecting this first part of that path gives a "bad" path from A. In fact there is a one-one correspondence between "bad" paths from A to P, and <u>all</u> paths from B to P. But the total number of paths from B to P is given by $^{p+q-1}C_{q-1}$. Hence $N = {}^{p+q-1}C_q - {}^{p+q-1}C_{q-1} = \frac{(p+q-1)!}{p!q!}(p-q)$ and finally $\frac{N}{T} = \frac{p-q}{p+q}$ as asserted.

Correct solutions received from: D. Stephens (Knox Grammar School)

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(Continued from page 19)

These six students are:

Adrian Chen: Prince Alfred College, S.A.

Carl Dettman: University High, Vic.

Peter Liu: St. Xavier, N.S.W.

Mitchell Porter: Towoomba Grammar, Qld.

James Stankovich: Melbourne Church of England Grammar, Vic.

Duncan Stephens: Knox Grammar, NSW.

In the next issue I will include some details of the Training School and further information on this year's contestants. If you wish to gain more information on the Mathematical Olympiad please contact

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