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SERIES

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Have you ever been asked to determine the next term in a series, given only the previous terms? Often with shorter series, I have found a number of simple ways in which the series might have been generated, but these can give different values for the next term. I used to wonder whether there was really any "correct" answer to such a problem. I have used fairly standard notation, referring to the series as $U_0, U_1, U_2 \ldots U_n$. A series can be dealt with more or less as a special kind of function by defining $U_n = f(n)$. I have found that most simple series fall into two groups. Clearly there will be many exceptions, but a compromise must be reached.

The first group includes all series defined by polynomial functions, including A.P.'s. ie. $U_n = \sum_{j=1}^{k} a_j n^j$. A broader group might include negative values of n, and even rationals. It is well known that there is a unique polynomial of at most n^{th} degree which passes through n+1 arbitrary points* (the n values must be distinct). If we define n+1 points to be (k, U_k) (from k=0 to k=n), the theorem implies that any series of n terms can be defined by a unique polynomial of at most n-1 degree. This answers the question I originally posed. Given n terms of a series, any value for the next term can be justified by a polynomial of at most nth degree.

The polynomial required is in theory easy to calculate, but in practice, this is best left to a computer. If we let $U_x = a_n x^n + a_{n-1} x^{n-1} \ldots + a_0$, we know that:

$$a_n . 0^n + a_{n-1} . 0^{n-1} + \ldots\ldots\ldots + a_0 = S_0$$

$$\vdots \qquad \vdots$$

$$a_n . n^n + a_{n-1} n^{n-1} + \ldots\ldots\ldots + a_0 = S_n$$

* see ALGEBRA by ARCHBOLD Chp.7.3.

The coefficients of a_i terms themselves form a polynomial series. Let the coefficient of $a_j = g_i(j) = b_n j^n + b_{n-1} j^{n-1} + \ldots + b_1 j + b_0$.

Now $g(j+k) - g(j) = (b_n(j+k)^n + b_{n-1}(j+k)^{n-1} + \ldots + b_1(j+k) + b_0)$

$- (b_n j^n + b_{n-1} j^{n-1} + \ldots + b_1 j + b_0)$

$= b_n[(j+k)^n - j^n] + b_{n-1}[(j+k)^{n-1} - j^{n-1}] \ldots\ldots + b_0(1-1)$

$= b_n[\binom{n}{1} j^{n-1} k + \binom{n}{2} j^{n-2} k^2 + \ldots + \binom{n}{n} k^n] + \ldots\ldots + 0$

In this polynomial every power of j is less than or equal to $n-1$. Thus the polynomial $g(j+k) - g(j)$ is of degree 1 less than $g(j)$. (Note that if $n=0$, then $g(j+k) = g(j) = b_0$, $g(j+k) - g(j) = 0$ - a null polynomial). By repeated subtraction, beginning with the term a_n, we can reduce the coefficient to a polynomial of zeroth degree, with all terms to the left known, and all those to the right having a zero coefficient. A polynomial of zeroth degree is a constant, and in the case where the polynomial equals x^n, the coefficient we arrive at is equal to $n!$. I have included a program which I have written, which calculates the necessary polynomial in this manner.

The second sort of series is of the general form S_n given by $S_n = \sum_{x=1}^{k} a_x r_x^{\,n} = a_1 r_1^n + a_2 r_2^n + \ldots + a_k r_k^n$. This clearly includes the G.P.: $S_n = ar^n$. Note also that $F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$ where F_n is the Fibonacci series. In fact suppose we define a series, G_n such that $k_0 G_n = k_1 G_{n-1} + k_2 G_{n-2} + \ldots k_j G_{n-j}$. If r is any root of the polynomial $k_0 r^j = k_1 r^{j-1} + k_j r^0$, then the series beginning $1, r, r^2, r^3, r^4, \ldots, r^j$ with each term r times the previous one satisfies the stated recurrence relation. A simple proof by induction is easily found. There are clearly j such roots (not necessarily distinct or real). Consider the equations

$$a_1 r_1^0 + a_2 r_2^0 + \ldots\ldots + a_k r_k^0 = S_0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_1 r_1^n + G_2 r_2^k + \ldots\ldots + a_k r_k^k = S_k$$

As the coefficients of a forms a series beginning $1, r, r^2, r^3, \ldots, r^k$, each coefficient of successive terms of a series G_n will continue to grow by a factor of r. Thus $U_n = a_1 r_1^n + a_2 r_2^n + \ldots\ldots + a_k r_k^n$ for each positive integer n and this gives the general solution of the recurrence (if $r_1, r_2, \ldots, r_k$ are distinct). It is interesting also to notice that with this type of series $\frac{U_n}{U_{n-1}}$ converges on the largest root of the polynomial if it is real and distinct, as $n \to \infty$. With the Fibonacci series, the limit is $\frac{\sqrt{5} + 1}{2}$, the golden mean.

I have not yet written a program to generate this sort of series given the first n terms. Perhaps some readers may wish to follow this up. I have experimented with the polynomial program allowing negative, and even fractional powers of x. Clearly there are a great number of ways in which a series may be defined, and there is great room for suggestion.

The computer program follows:-

```
1Ø PRINT "ENTER SERIES:" :PRINT "TYPE END WHEN SERIES IS FINISHED."
2Ø DIM P(42), S(42), Q(42), E(42), F(42)
3Ø P(Ø) = 1: S(Ø) = 1
4Ø N=N+1 :  INPUT A$: Q(N) = VAL(A$)
5Ø P(N) = P(N-1)*N : S(N) = - S(N-1)
6Ø IF A$ >< "END" THEN 4Ø
7Ø N=N + 1
8Ø FOR Y=N TO 1 STEP -1
9Ø FOR Z=Y TO 1 STEP -1
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Set up and enter series

```
100 E(Y)=E(Y) + Q(Z)*P(Y-1)/P(Y-Z)/P(Z-1)*S(Z-1)
110 NEXT Z: IF N=Y THEN Z=Y : GO TO 140
120 FOR Z=N TO Y+1 STEP -1 = GOSUB4
130 E(Y) = E(Y) - E(Z)*A: NEXT Z
140 GOSUB170: E(Y) = E(Y)/A
150 F(Y) = E(Y) + .0005: F(Y) = INT(F(Y)*1000)/1000
160 NEXT Y = GO TO 200
170 A=0: FOR X=Y TO 1 STEP -1
180 A=A + INT(X↑(Z-1)*P(Y-1)/P(Y-X)/P(X-1))*S(X-1)
190 NEXT X: RETURN
200 REM PRINT UP
210 PRINT: PRINT "Y=";: V=0
220 FOR A=N TO 1 STEP -1
230 IF F(A) = 0 THEN 300
240 IF F(A) > 0 AND V=1 THEN PRINT "T";
250 V=1: IF F(A) <> 1 THEN PRINT F(A);
260 G=A -1 : IF G = < 0 AND F(A) = 1 THEN PRINT "/";
270 IF G < 0 THEN PRINT "/";
280 IF G >< 0 THEN PRINT "X";
290 IF ABS(G) > 1 THEN PRINT "↑"; ABS(G);
300 NEXT A: IF V=0 THEN PRINT "0";
310 PRINT: PRINT: PRINT "THE NEXT TERMS IN THIS SERIES ARE:"
320 W=N+1
330 A=0: FOR X = 1 TO N: A = A + E(X)*W↑(X-1): NEXT X
340 A=A + .0000 5: A = INT(A * 10000)/10000
350 PRINT A: W = W+1: GO TO 330
```

Calculate series (lines 100–140)

ROUND OFF (line 150)

Subroutine calculating coefficient of a_y (lines 170–190)

Print polynomial in x (lines 200–300)

Print up successive terms (lines 310–350)

Note $P(N) = N!$ $S(N) = (-1)^N$

$Q(N)$ = The nth term of the series

$E(N)$ = The coefficient of X^{n-1} in the polynomial formula

$F(N) = E(N)$ rounded off to 3 decimal places.

N is the variable containing the power of the X in the polynomial formula, for the current coefficient.