Parabola Volume 21, Issue 2 (1985)

THE ROOT OF THE PROBLEM

BY

WILLIAM G. TAYLOR

There have been many methods proposed for finding square roots. The easiest is to use the button on the calculator, but this has limited accuracy and is difficult to represent as a rational number.

A simple and efficient algorithm for determining rational approximations to square roots, to any required accuracy, is a useful tool in many respects:

- for the teacher attempting to redress the omnipotence of the calculator in the eyes of the pupils;
- for the purist to help complete a repertoire of mathematical tools;
- for those times when the battery is flat or the decimal places are insufficient.

This algorithm is based on Euclid's algorithm and more specifically on the theory of continued fractions and a "magic" table. Its attraction is its simplicity.

First, I will present the concept of continued fractions.

Second - simplification of this, called a "magic" table.

Third - how to go about finding a square root using the above process.

Fourth - a touch of algebra to simplify the third process.

Fifth - a simple method of calculating square roots.

Sixth - even if none of the other five steps make sense, you can still use this method.

First - Continued Fractions

This process merely expresses improper fractions as a whole number plus (or minus) a remainder. But the remainder is expressed as "one over its reciprocal" which allows the process to be repeated.

Consider the fraction $\frac{13}{7}$

Now (A) $\frac{13}{7} = 1 + \frac{6}{7}$ but we will now express this in a different way.

(B) $\frac{13}{7} = 1 + \frac{1}{7/6}$ Note that $\frac{"7"}{6}$ is an improper fraction and that

(C) $\frac{7}{6} = 1 + \frac{1}{6}$ and now we replace $\frac{7}{6}$ in equation (B) with $1 + \frac{1}{6}$

so we get:

(D) $\frac{13}{7} = 1 + \cfrac{1}{1 + \cfrac{1}{6}}$ The process terminates when the numerator of the remainder is one as in equation (C).

Here is a more streamlined and longer example:

$$\frac{33}{14} = 2 + \frac{1}{14/5} \qquad \left(\frac{14}{5} = 2 + \frac{1}{4/5}\right)$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{5/4}} \qquad \left(\frac{5}{4} = 1 + \frac{1}{4/1}\right)$$

End of process since $4/1$ is an integer

$$= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4}}}$$

This final step is written $\frac{5}{4} = 1 + \frac{1}{4}$

You can see how the term "continued fraction" occurs.

We call the set of whole number parts in a continued fraction its partial quotients and write them in square brackets.

E.g. the set of partial quotients for $\frac{13}{7} = [1,1,6]$

and for $\frac{33}{14} = [2,2,1,4]$

You can read them from the partial fraction or by using a neater procedure. We will only need the partial quotients so let's throw out the unwieldy continued fraction expression getting the partial quotients for $\frac{33}{14}$ more neatly.

$$\frac{33}{14} = 2 + \frac{5}{14}$$

$$\frac{14}{5} = 2 + \frac{4}{5}$$

$$\frac{5}{4} = 1 + \frac{1}{4}$$

$$4 = 4 + 0$$

read the partial quotients from top to bottom [2,2,1,4]

<u>Second - The "magic" Table</u>

This is a process to regenerate a fraction given its set of partial quotients. Here is a magic table for the partial quotients [2,2,1,4] and a description of how to fill it in.

One column for each partial quotient			2	2	1	4
Row A	0	1	0+1x2 = 2	1+2x2 = 5	2+5x1 = 7	5+7x4 = 33
Row B	1	0	1+0x2 = 1	0+1x2 = 2	1+2x1 = 3	2+3x4 = 14

always begin with this pattern - work from left to right

Each entry is calculated by multiplying the number at the top of a column by the number in the same row and the column one to the left and adding the number in the same row and the column two to the left.

Here is the table again without the working:

			2	2	1	4 ....
Row A	0	1	2	5	7	33 ...
Row B	1	0	1	2	3	14
			↓	↓	↓	↓

Can you see the fraction $\frac{33}{14}$ in there?

Now $\frac{A}{B}$ $\frac{2}{1}$ $\frac{5}{2}$ $\frac{7}{3}$ $\frac{33}{14}$ ....

If you plot these fractions on a number line, you will see that they jump from one side of $\frac{33}{14}$ to the other and get closer with each jump. They are all approximations for $\frac{33}{14}$ and in fact are optimal in the sense of being the closest to $\frac{33}{14}$ for the least work.

Third - Square Roots.

The partial quotients and magic table can be used for getting "optimal" approximations for square roots.too. Take $\sqrt{7}$ for example. First we write it as a fraction, and that's easy : $\frac{\sqrt{7}}{1}$

Next we find its partial quotients (remember $\sqrt{7}$ is between 2 & 3).

$$\frac{\sqrt{7}}{1} = 2 + \frac{\sqrt{7}-2}{1} \quad *$$

The next line requires us to use:

$$\left(\frac{1}{\sqrt{7}-2}\right) \text{ which is } \frac{\sqrt{7}+2}{\sqrt{7}+2} \times \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{3}$$

Note $\frac{\sqrt{7}+2}{3}$ is between $\frac{2+2}{3}$ and $\frac{3+2}{3}$, i.e. roughly 1.

$$\frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3} \qquad \frac{3}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{2}$$ which is between 1 & 2.

$$\frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2} \qquad \frac{2}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{3}$$

$$\frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3} \qquad \frac{3}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{1}$$

$\frac{\sqrt{7} + 2}{1} = 4 + \frac{\sqrt{7} - 2}{1}$ *

Notice these remainders at * are the same and so the process will repeat.

The set of partial quotients for $\sqrt{7}$ is:

[2,1,1,1,4,1,1,1,4,1,1,1,4,....] Yes, it's infinite but it repeats. In fact, it repeats for all square roots (but not for cube roots or higher, as far as we know).

Magic table for $\sqrt{7}$ partial quotients $[2; \overline{1,1,1,4}]$

		2	1	1	1	4	1	1	1	4	...
0	1	2	3	5	8	37	45	82	127	590	....
1	0	1	1	2	3	14	17	31	48	223	....
		↓	↓	↓	↓	↓	↓	↓	↓	↓	
Approximations for $\sqrt{7}$		$\frac{2}{1}$	$\frac{3}{1}$	$\frac{5}{2}$	$\frac{8}{3}$	$\frac{37}{14}$	$\frac{45}{17}$	$\frac{82}{31}$	$\frac{127}{48}$	$\frac{590}{223}$	....

in fact $\frac{45}{17}$ is less than .0014 different from $\sqrt{7}$ and if you go further, the error gets less and less; $\frac{590}{223}$ has an error of only .0000114.

<u>Fourth - Make it Simpler</u>

This procedure works for all square roots but it is not very useful for several reasons:

(1) It takes a lot of work to get the partial quotients

(2) The repeating pattern varies in length.

It would be good if these two problems could be eliminated and fortunately they can.

You can determine the partial quotients by a simple calculation and the repeating pattern is never more than two numbers long provided you allow the partial quotients to be fractions themselves.

Here is a bit of algebra that helps resolve the problem. Suppose we want to find $\sqrt{x}$ where x is some number. Let n^2 be the closest perfect square to x:

Let $n^2 + a = x$, then $1 - 2n < a < 2n + 1$

This is easily verified because $(n-1)^2 < n^2 + a < (n+1)^2$

For example : $\sqrt{7} = \sqrt{3^2 - 2}$ or $\sqrt{2^2 + 3}$.

Using our procedure for finding partial quotients: (closest whole number to $\sqrt{n^2 + a}$ is n)

$$\sqrt{n^2 + a} = n + (\sqrt{n^2 + a} - n)^*$$

Note: $\dfrac{1}{\sqrt{n^2 + a} - n} = \dfrac{\sqrt{n^2 + a} + n}{a}$

$$\frac{\sqrt{n^2 + a} + n}{a} = \frac{2n}{a} + \left(\frac{\sqrt{n^2 + a} - n}{a}\right)$$

Note: $\dfrac{a}{\sqrt{n^2 + a} - n} = \dfrac{\sqrt{n^2 + a} + n}{1}$

$$\frac{\sqrt{n^2 + a} + n}{1} = 2n + (\sqrt{n^2 + a} - n)^*$$

*The same remainder means the pattern repeats

So the set of partial quotients for $\sqrt{x}$ is:

$$[\, n; \overline{\tfrac{2n}{a}, 2n}\,]$$

where n^2 is a perfect square nearest to but on eitherside of x and $a = x - n^2$.

Notice that if "a" does not divide 2n, then $\frac{2n}{a}$ is itself a fraction but this will not hinder us too much in using the magic table.

Fifth - The Simple Method

Once again let us calculate $\sqrt{7}$.

Let us taken $n = 2$ and hence $a = 3$. The partial quotients are thus $[2; \overline{\tfrac{4}{3}, 4}]$.

That was a lot easier, wasn't it !

		2	$\frac{4}{3}$	4	$\frac{4}{3}$	4	.....
0	1	2	$\frac{11}{3}$	$\frac{50}{3}$	$\frac{233}{9}$	$\frac{982}{9}$	.....
1	0	1	$\frac{4}{3}$	$\frac{19}{3}$	$\frac{88}{9}$	$\frac{371}{9}$	.....
		↓	↓	↓	↓	↓	
		$\frac{2}{1}$	$\frac{11}{4}$	$\frac{50}{19}$	$\frac{233}{88}$	$\frac{982}{371}$	

The approximation $\frac{982}{371}$ compares favourably with $\frac{47}{17}$ from the other magic table approximation. The error is around .0014 and it is generated after five instead of six steps but the numbers are larger and in that sense it is not optimal.

We now have a very quick way of finding partial quotients and the magic table is easy to use in generating apprxomations for square roots to any accuracy.

One last example $\surd 29$ $x = 29$, $(29 = 5^2 + 4)$ and so $n = 5$ and $a = 4$

partial quotients $[n \; ; \; \frac{2n}{a} \, , \; 2n] = [5 \; ; \; \frac{5}{2} \, , \; 10 \,]$

		5	$\frac{5}{2}$	10	$\frac{5}{2}$	
0	1	5	$\frac{27}{2}$	140	$\frac{727}{2}$	
1	0	1	$\frac{5}{2}$	26	$\frac{135}{2}$	
		↓	↓	↓	↓	
		$\frac{5}{1}$	$\frac{27}{2}$	$\frac{140}{26}$	$\frac{727}{135}$	
				↓	|	
				$\frac{70}{13}$		→ error ≃ .00002

Comment: Readers who would like to know more about the magic of continued fractions might consult A. KHINCHIN, "Continued factrions." (University of Chicago Press, 1964).