THE 1985 SCHOOL MATHEMATICS COMPETITION

JUNIOR DIVISION

1. Find all 4 digit numbers, abcd, such that when the second digit (b) is deleted the remaining 3 digit number acd is a factor of the original number.

SOLUTION

If the number "acd" is: then "abcd" is one of:

100	1000,1100,1200,1300,...,1900
200	2000,2200,2400,2600,2800
300	3000,3300,3600,3900
400	4000,4400,4800
500	5000,5500
600,700	6000,6600,7000,7700
800,900	8000,8800,9000,9900

If "acd" is: then "abcd" is one of:

150	1050,1350,1650,1950
250	2250,2750
350	3150,3850
450	4050,4950

Now to understand the general case suppose "abcd" = s."acd". Then

$5 < s < 20$ since $5.199 < 1000$ and $20.101 > 2000$.

Also 100 divides "abcd" - "acd" thus (s-1)."cd" is a multiple of 100. Hence "cd" = 00,50,25,75,20,40,60,80,10,30,70 or 90, since in all other cases s would have to be greater than 20. Hence the remaining solutions are:

(125;1125,1625) (225;2025,2925)(175;1575)(275;2475)(375;3375)(475;4275)(575;5175)(675;6075)

(120;1320,1920)(220;2420)(320;3520)(420;4620)(520;5720)(620;6820)(720;7920)

(140;1540)(240;2640)(340;3740)(440;4840)(540;5940)

(160;1760)(260;2860)(360;3960)(180;1080,1980)

(110;1210)(210;2310)...(810;8910)

(130;1430)(230;2530)(330;3630)(430;4730)(530;5830)(630;6930)

(170;1870)(270;2970)

This gives a total of 86 numbers with the required property.

2. Are there integers a,b which satisfy

$$5a^2 - 7b^2 = 9?$$

Either find them or show that they do not exist.

<u>SOLUTION</u>

If $5a^2 - 7b^2 = 9$ then 5 does not divide b hence the remainder on dividing b by 5 is 1,2,3 or 4.

i.e. $b = 5c + d$, $d = 1,2,3$ or 4.

$\therefore$ $b^2 = (25c^2 + 10cd) + d^2$ with $d^2 = 1,4,9$ or 16

$\therefore$ $9 = 5a^2 - 7b^2 = 5(a^2 - 7.5c^2 - 7.2cd) - 7,28,63$ or 112

and 16,27,72 or 121 $= 5(a^2 - 35c^2 - 14cd)$ which is not possible

i.e. no such b exists.

<u>More concisely</u> $5a^2 \equiv 0 \pmod 5$

$7b^2 \equiv 7.1$ or $7.4 \equiv 2$ or $3 \pmod 5$

$\therefore$ $5a^2 - 7b^2 \equiv -2$ or $-3 \equiv 3$ or $2 \pmod 5$

but $9 \equiv 4 \pmod 5$ $\therefore$ No solutions.

3. Find all solutions of the equations

$$u^3 + v^8 = 1984,$$

$$x^3 + y^6 + z^8 = 1985,$$

where u, v, x, y, z are positive integers.

Hence or otherwise find a solution to $a^3 + b^3 = c^3 + d^3$ with a,b,c and d different positive integers.

SOLUTION

$1^8 = 1,\ 2^8 = 256,\ 3^8 = 6561$

$\therefore\ v = 1$ or 2

$$1984 - 1 = 1983 \neq x^3$$

$$1984 - 256 = 1728 = 12^3$$

$\therefore\ a = 12\quad v = 2$ is the only solution.

Now $z = 1$ or 2

$1^6 = 1,\ 2^6 = 64,\ 3^6 = 729,\ 4^6 = 4096$

$\therefore\ y = 1, 2$ or 3

$1985 - 1^8 - 1^6 = 1983 \neq x^3$

$1985 - 1^8 - 2^6 = 1920 \neq x^3$

$1985 - 1^8 - 3^6 = 1255 \neq x^3$

$1985 - 2^8 - 1^6 = 1728 = 12^3$

$1985 - 2^8 - 2^6 = 1665 \neq x^3$

$1985 - 2^8 - 3^6 = 1000 = 10^3$

$\therefore\ 12^3 + 2^8 = 1984$ and

$12^3 + 2^8 + 1^6 = 10^3 + 3^6 + 2^8 = 1985$ are the only solutions.

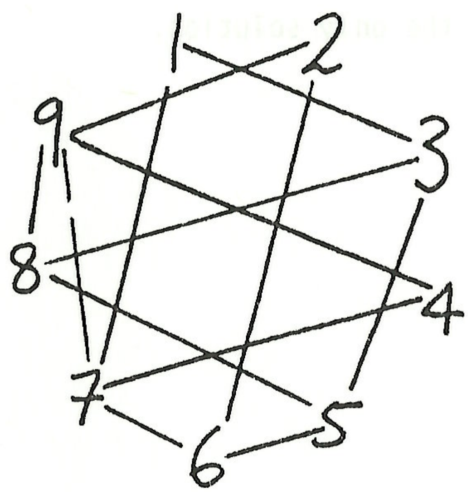
From the above

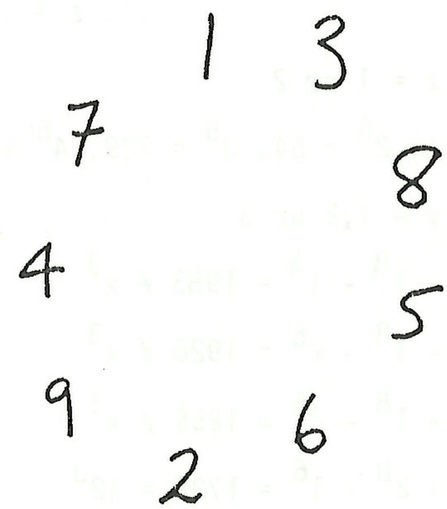
$$12^3 + 1^6 = 10^3 + 3^6$$

$$\therefore\ \boxed{12^3 + 1^3 = 10^3 + 9^3} = 1729.$$

4. Write the numbers $1,2,3,\ldots,9$ in a big circle. Join the pair m,n if and only if $m + n$ is not divisible by $3,5$ or 7. Use the resulting diagram (or any other method) to write the numbers $1,2,3,\ldots,9$ in <u>some</u> order on a circle so that the sum of each pair of neighbours is not divisible by $3,5$, or 7.

SOLUTION





5. At a party of 21 people each person knows at most four others. Prove that there are five in the party who mutually do not know each other.

SOLUTION

Suppose the people are all called John or Jill so that we may conveniently label them $J_1, J_2, J_3, \ldots, J_{21}$.

Now J_1 knows at most 4 people, hence by renumbering if necessary, J_1 does not know $J_6, J_7, \ldots, J_{21}$. J_6 knows at most 4 of $J_2, J_3, J_4, J_5, J_7, J_8, \ldots, J_{21}$, hence by renumbering if necessary J_6 knows at most 4 of $J_2, J_3, J_4, J_5, J_7, J_8, J_9, J_{10}$ and both J_1 and J_6 do not know $J_{11}, J_{12}, \ldots, J_{21}$.

Similarly we may assume J_1, J_6 and J_{11} do not know $J_{16}, J_{17}, J_{18}, J_{19}, J_{20}$ and J_{21}.

Finally J_{16} does not know at least one of $J_{17}, \ldots, J_{21}$.

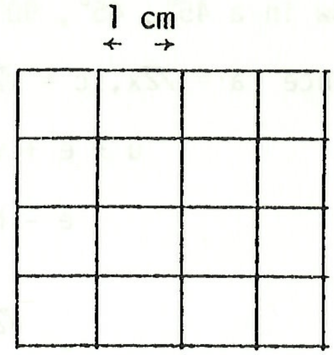
Hence by renumbering if necessary $J_1, J_6, J_{11}, J_{16}, J_{21}$ do not know each other.

6. Can one make a 4×4 square lattice of size 4 cm by 4 cm by using

(a) 5 pieces of thread, each 8 cm long?

(b) 8 pieces of thread, each 5 cm long?

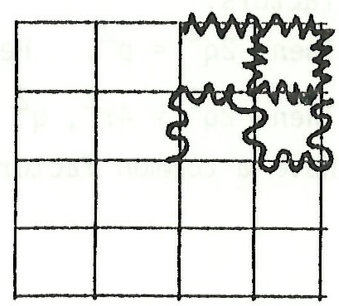
Give reasons.



SOLUTION

One may regard the lattice as a graph consisting of 25 vertices and 40 edges (each of length 1cm). The graph has 4 vertices of order 2, 12 vertices of order 3 and 9 vertices of order 4, where the order of a vertex is the number of edges meeting at a vertex.

If a vertex has odd order then the end of a thread must be placed there - hence we need at least 6 pieces of thread to make the lattice.



With 8 of length 5 cm there are various solutions e.g. 

and 

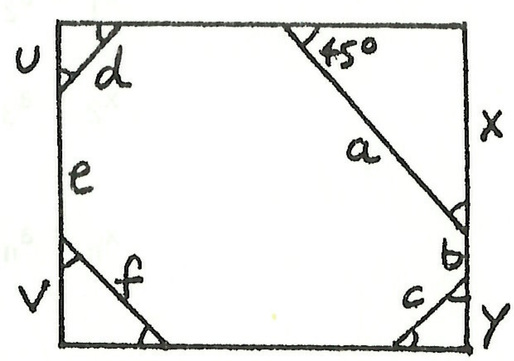
and the 6 sections constructed by rotating by 90°, 180° and 270°.

7. Prove that there is no convex eight sided polygon with all angles equal and the sides distinct integers.

SOLUTION

A convex 8 sided polygon with all angles equal has all angles $= 180 - \frac{360}{8} = 135°$

Hence the polygon can be fitted into a rectangle as shown. Let the lengths of the sides be a,b,c,d,e,f as shown all integers.



Now in a 45°, 45°, 90° triangle the ratio of the sides are $1 : 1 : \sqrt{2}$

Hence $a = \sqrt{2}x$, $c = \sqrt{2}y$, $f = \sqrt{2}v$, $d = \sqrt{2}u$

$$u + e + v = x + b + y$$

$$e - b = x + y - u - v = \frac{a+c-f-d}{\sqrt{2}}$$

$$\sqrt{2} = \frac{a+c-f-d}{e-b} \qquad e \neq b$$

$$= p/q \qquad p,q \text{ positive integers.}$$

This says that $\sqrt{2}$ is rational which is not true - hence there is no such octagon.

[<u>THEOREM</u> $\sqrt{2}$ is irrational.

<u>PROOF</u> Suppose $\sqrt{2} = \frac{p}{q}$ with p and q integers with no common factors.
Then $2q^2 = p^2$. Hence p is even, say $p = 2r$.
Then $2q^2 = 4r^2$, $q^2 = 2r^2$ thus q is even and we have shown p and q have a common factor. This contradiction shows that $\sqrt{2}$ is irrational.]

SENIOR DIVISION

1. (i) Given that $a_1, a_2, \ldots, a_n, \ldots$ are positive integers chosen as large as possible and that $x_1, x_2, \ldots, x_n, \ldots$ are positive real numbers, find the sequence (a_n) such that

$$\sqrt{6} = a_1 + x_1^{-1}$$

$$x_1 = a_2 + x_2^{-1}$$

$$x_2 = a_3 + x_3^{-1}$$

$$\vdots$$

$$x_n = a_{n+1} + x_{n+1}^{-1}$$

$$\vdots$$

(ii) Which number corresponds to the sequence

$$3,3,6,3,6,\ldots,3,6,3,6,\ldots$$

in the same way that $\sqrt{6}$ corresponds to the sequence (a_n) given in (i)?

SOLUTION

(i) $\sqrt{6} = 2 + x_1^{-1}$, $x_1 = 2 + x_2^{-1}$, $x_2 = 4 + x_3^{-1}$, $x_3 = 2 + x_4^{-1}$, $x_4 = 4 + x_5^{-1}$, $x_5 = 2 + x_6^{-1}, \ldots$.

Hence the pattern appears to be

$(2;\ 2,4,2,4,2,4,\ldots)$

Now $x_1 = 1/(\sqrt{6}-2) = (\sqrt{6}+2)/2$

$x_2 = 1/(\tfrac{1}{2}(\sqrt{6}+2)-2) = 2/(\sqrt{6}-2) = \sqrt{6} + 2$

$x_3 = 1/(\sqrt{6}+2-4) = 1/(\sqrt{6}-2) \qquad \therefore\ x_1 = x_3$

and the pattern is as it appears.

(ii) Suppose $a = 3,3,6,3,6,3,6,\ldots$ then $x_1 = x_3$ for this a.

$a = 3 + x_1^{-1}$, $x_1 = 3 + x_2^{-1}$, $x_2 = 6 + x_3^{-1} = 6 + x_1^{-1}$

Hence $(x_1-3)(6+x_1^{-1}) = 1$

$(x_1-3)(6x_1+1) = x_1$

$6x_1^2 - 18x_1 - 3 = 0$

$2x_1^2 - 6x_1 - 1 = 0$

$x_1 = (6\pm\sqrt{36+8})/4 = (3\pm\sqrt{11})/2$

and $x_1 = \tfrac{1}{2}(3+\sqrt{11})$ since $x_1 > 0$.

Hence $a = 3 + 2/(3+\sqrt{11}) = 3 + \sqrt{11} - 3 = \sqrt{11}$.

2. Find all solutions of the system of equations

$$x = \tfrac{1}{2}(y + \tfrac{1}{y})$$

$$y = \tfrac{1}{2}(z + \tfrac{1}{z})$$

$$z = \tfrac{1}{2}(t + \tfrac{1}{t})$$

(continued over)

$$t = \tfrac{1}{2}(x + \frac{1}{x}).$$

Generalize to 1985 variables.

SOLUTION

Suppose $x > 0$. Since $t = \frac{1}{2}(x + \frac{1}{x})$ is the arithmetic mean of x and $\frac{1}{x}$, t lies between x and $\frac{1}{x}$, and so therefore does $\frac{1}{t}$. Similarly, from $z = \frac{1}{2}(t + \frac{1}{t})$, z and $\frac{1}{z}$ lie between t and $\frac{1}{t}$. Continuing, y and $\frac{1}{y}$ lie between z and $\frac{1}{z}$, and x and $\frac{1}{x}$ lie between y and $\frac{1}{y}$. Here, between means strictly between unless $x = t = z = y = 1$. So the only solution with $x > 0$ is $x = t = z = y = 1$. Similarly the only solution with $x < 0$ is $x = t = z = y = -1$.

If there are 1985 variables $x_1, \ldots, x_{1985}$ with $x_i = \frac{1}{2}(x_{i-1} + \frac{1}{x_{i-1}})$ $i = 2, \ldots, 1985$ and $x_1 = \frac{1}{2}(x_{1985} + \frac{1}{x_{1985}})$ then the only solustion is $\underset{\sim}{x} = (1,1,\ldots,1)$ or $(-1,-1,-1,\ldots,-1)$.

3. A box contains p white balls and q black balls. Beside the box there is a pile of black balls. Two balls are taken out of the box. If they are of the same colour, a black ball from the pile is put into the box. If they are of different colours, the white ball is put back into the box. This procedure is repeated until the last pair of balls is removed from the box and one last ball is put in. What is the probability that this last ball is white?

SOLUTION

p white q black $\xrightarrow{WW}$ $p-2$ white $q+1$ black

$\xrightarrow{BW}$ p white $q-1$ black

$\xrightarrow{BB}$ p white $q-1$ black

Hence after each step the parity (odd or even) of the number of white balls in unchanged but the total number of balls is reduced by 1.

(continued over)

Hence if p is odd the last ball is white

if p is even the last ball is black, (with probability 1 in both cases).

4. ABC is a triangle, right angled at C. Let CD be perpendicular to AB. The bisector of $< CDB$ meets CB in X and the bisector of $< ADC$ meets AC in Y. Prove that CX = CY.

SOLUTION

If $\Delta ABC ||| \Delta DEF$ then $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

where $|||$ means is similar to.

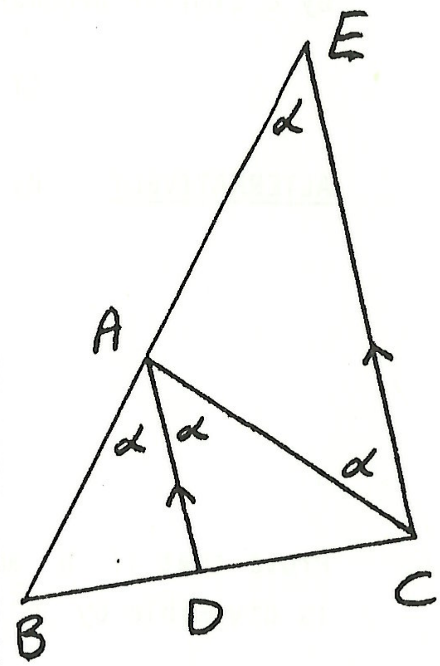
If AD is the angle bisector of angle A in ΔABC then $\frac{BA}{AC} = \frac{BD}{DC}$.

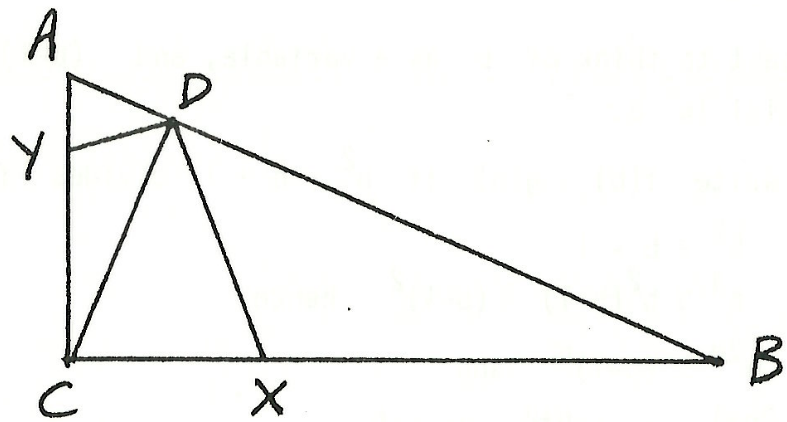
Proof. Extend BA to meet the line through C parallel to AD in E.

Now $< BAD = < AEC = \alpha$ (corresponding angles) and $< DAC = < ACE = \alpha$ (alternate angles).

Hence ΔAEC is isoceles and AE = AC.

Thus $\frac{BD}{DC} = \frac{BA}{AE} = \frac{BA}{AC}$ as required.



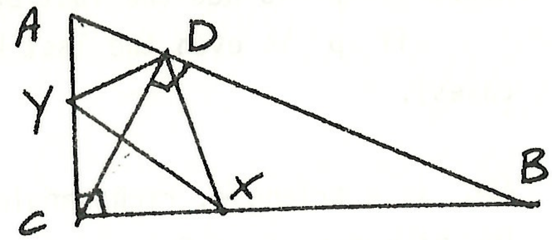


$\Delta ABC ||| \Delta CBD ||| \Delta ACD$

$\therefore \frac{AB}{CB} = \frac{AC}{CD} = \frac{BC}{BD}$ and $\frac{CB}{AC} = \frac{CD}{AC} = \frac{BD}{CD}$

(continued over)

Also $\frac{CD}{DB} = \frac{CX}{XB} = \frac{CX}{CB-CX}$



Thus $CD.CB - CD.CX = DB.CX$

$$CX = \frac{CD.CB}{CD+DB}$$

$$= \frac{CD}{\frac{CD}{CB} + \frac{BD}{CB}} = \frac{CD}{\frac{AC}{AB} + \frac{CB}{AB}} = \frac{AB.CD}{AC+CB}$$

By a similar argument

$$CY = \frac{BA.CD}{BC+CA}, \quad \text{thus} \quad CX = CY.$$

<u>ALTERNATIVELY</u> By data, $\angle CDX = \angle CDY = 45°$. Thus XY subtends right angles at C and D and CXDY is a cyclic quadrilateral. Now $\angle CYX = \angle CDX = 45°$ (subtended by CX) and $\angle CXY = \angle CDY = 45°$ (subtended by CY). So ΔCXY is isosceles and $CX = CY$.

5. Prove that if b and n are positive integers then $(b-1)^{n+2} + b^{2n+1}$ is divisible by $b^2 - b + 1$.

<u>SOLUTION</u>

It is best to think of b as a variable, and $(b-1)^{n+2} + b^{2n+1}$ as a polynomial in b.

Let us write $f(b) \equiv g(b)$ if $b^2 - b + 1$ divides $f(b) - g(b)$.

Now $b^2 \equiv b - 1$

Thus $b^4 \equiv b^2(b-1) \equiv (b-1)^2$ hence

$$b^{2n} \equiv (b-1)^n \quad \text{and}$$

$$(b-1)^{n+2} + b^{2n+1} \equiv (b-1)^{n+2} + (b-1)^n b$$

$$= (b-1)^n(b^2-2b+1+b)$$

$$= (b-1)^n(b^2-b+1)$$

and we have shown $(b-1)^{n+2} + b^{2n+1}$ differs from a multiple of $b^2 - b + 1$ by a multiple of $b^2 - b + 1$ and the result is proved.

6. Given 16 distinct positive integers $a_1, a_2, \ldots, a_{16}$, each less than or equal to 100, prove that there are four distinct ones among them, a_i, a_j, a_k, a_m, such that $a_i + a_j = a_k + a_m$.

SOLUTION

Suppose $1 \leqslant a_1 < \ldots < a_{16} \leqslant 100$. The number of pairs (a_i, a_j) with $i > j$ is $\binom{16}{2} = 120$.

The differences $a_i - a_j$ with $i > j$ range between 1 and 99, i.e. 99 possible values. So there are at least 21 cases of equality

$$a_i - a_j = a_k - a_\ell \quad \text{with} \quad i > j,\ k > \ell.$$

Suppose there are no such equations with i,j,k,ℓ distinct. Then the cases of equality must all be of the shape

$$a_i - a_j = a_j - a_k \quad \text{with} \quad 16 \geqslant i > j > k \geqslant 1.$$

There are 14 possible values of j and 21 such equations, so some j occurs twice, say

$$a_i - a_j = a_j - a_k,\ a_r - a_j = a_j - a_s \quad \text{with } i,j,k,r,s \text{ distinct.}$$

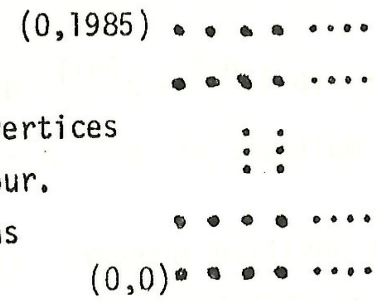
Subtracting gives $a_i - a_r = a_s - a_\ell$ with i,r,s,ℓ distinct, contrary to hypothesis.

So there are 4 distinct a's with $a_i - a_k = a_m - a_j$, i.e. $a_i + a_j = a_k + a_m$, as required.

7. Points in the plane with integer coordinates are called lattice points. We colour each lattice point with one of 1985 different colours. Prove that there are four lattice points of the same colour which form a rectangle with sides parallel to the coordinate axes.

SOLUTION

Consider a long strip of width 1986 lattice points starting at (0,0), (0,1985)



The first column contains 1986 vertices thus 2 are painted the same colour.

Similary each coloumn contains a pair of points painted the same colour.

Now the pair of points can be in any one of $\binom{1986}{2} = (1986.1985)/2$ "positions", (i.e. 1st and 2nd, 1st and 3rd etc.)

Hence after one has coloured $1985.\binom{1986}{2} + 1$ columns, 2 of the coloumns must each have a pair painted the same colour in the same "position". You now have a rectangle with sides parallel to the axes with its 4 vertices the same colour.

[This proof is an application of the Pigeon Hole Principle: if n + 1 objects are placed in n pigeon holes, at least one pigeon hole contains 2 objects.]

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