SOLUTIONS TO PROBLEMS FROM VOLUME 20, NUMBER 3

Q.612. From a point P inside a cube, line segments are drawn to each of the eight vertices of the cube, forming six pyramids each having P as the apex and a face of the cube as the base. Is it possible to place P in such a position that the volumes of the pyramids are in the ratio 1:2:3:4:5:6 ?

Answer: Since the volume of a pyramid is $\frac{1}{3}$ Area of base x perpendicular height, and for all six pyramids the base area is the same, we require a point P whose perpendicular distances from the 6 faces of the cube are in the given ratios. Such is a point inside the cube whose distances from the three faces meeting at one vertex are $\frac{1}{7}\ell$, $\frac{2}{7}\ell$, and $\frac{3}{7}\ell$ (where ℓ denotes the length of a side of the cube), since its distances from the opposite faces are respectively $\frac{6}{7}\ell$, $\frac{5}{7}\ell$, and $\frac{4}{7}\ell$.

Q.613. Find all pairs of integers x,y such that

$$5x^2 + 5xy + 5y^2 = 7x + 14y.$$

Answer: $5x^2 + 5xy + 5y^2 = 7x + 14y$

$\Rightarrow 100x^2 + 100xy + 100y^2 = 140(x + 2y)$

$\Rightarrow 25(x + 2y)^2 - 2 \times 14 \times [5(x + 2y)] + 75x^2 = 0$

$\Rightarrow (5(x + 2y) - 14)^2 + 75x^2 = 14^2.$

Since neither term on the L.H.S. an be negative, $75x^2 \leq 196$, so that x, if integral, can have no value other than −1, 0, or + 1.

Substituting these in turn into the original equation yields

$x = -1;\quad 5y^2 - 19y + 12 = 0 \Rightarrow y = 3$, (or $y = \frac{4}{5}$, not an integer)

$x = 0;\quad 5y^2 - 14y = 0 \Rightarrow y = 0$, (or $y = \frac{14}{5}$)

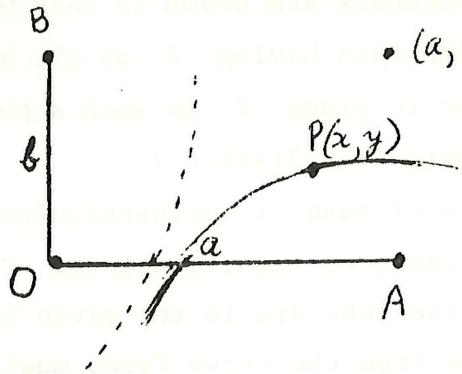
$x = 1;\quad 5y^2 - 9y - 2 = 0 \Rightarrow y = 2$, (or $-\frac{1}{5}$)

Thus the only solutions in integers are

$$(x,y) = (-1,3);\ (0,0);\ \text{or}\ (1,2).$$

Q.614. ∇OAB is rightangled at O, and P is a point in the plane of the triangle such that a rightangled triangle can be constructed with sides equal in length to OP, AP, and BP. Find the locus of P.

Answer: Take co-ordinate axes OA and OB, and let P have co-ordinates (x,y).



Then OP^2, BP^2, and AP^2 are respectively $x^2 + y^2$, $x^2 + (y - b)^2$, and $(x - a)^2 + y^2$. A right angled triangle can be formed from the lengths OP, BP, AP if and only if one of these squares (the hypotenu e) is the sum of the other two. There are three cases:

I. $x^2 + y^2 = (x^2 + (y - b)^2) + (x - a)^2 + y^2$

$$0 = (y - b)^2 + (x - a)^2$$

$(x,y) = (a,b)$ A single point on the locus results from this case.

II. $x^2 + (y - b)^2 = x^2 + y^2 + (x - a)^2 + y^2$

$$2b^2 = (x - a)^2 + y^2 + 2by + b^2$$

$$(\sqrt{2}b)^2 = (x - a)^2 + (y + b)^2$$ This yields points on a circle centre $(a,-b)$, radius $\sqrt{2}b$.

III. $(x - a)^2 + y^2 = (x^2 + y^2) + x^2 + (y - b)^2$ Similar to II and yields the circle centre $(-a,b)$, radius $\sqrt{2}a$.

Thus the locus consists of 2 circles and a single point. The circles do not intersect unless $a = b$ when they touch each other at the point O. In this case $(a = b)$ the right angled triangle corresponding to taking P at O collapses to one with a side of zero length. It is perhaps a matter of taste whether one should exclude this point from the locus.

Q.615. Let f be a function satisfying the functional equation

$$f(s,t) = f(2s + 2t, 2t - 2s)$$

for all real numbers s and t. Define g by $g(x) = f(2x,0)$. Prove that $g(x)$ is a periodic function. [i.e. Show that for some real number p, $g(x + p) = g(x)$ for every x.]

Answer: Apologies. The function $g(x)$ was intended to be defined by $g(x) =$ $g(x) = f(2^x,0)$. The solution of the corrected problem follows:

$$g(x) = f(2^x,0) = f(2^{x+1},-2^{x+1}) = f(0,-2^{x+3}) = f(-2^{x+4},-2^{x+4})$$

$$= f(-2^{x+6},0) = f(-2^{x+7},2^{x+7}) = f(0,2^{x+9}) = f(2^{x+10},2^{x+10})$$

$$= f(2^{x+12},0) = g(x + 12), \text{ for any } x.$$

(We have of course used the functional equation satisfied by f eight times). Hence g is periodic (with period $p = 12$).

Q.616.

1	2	3	4	5	.	.	.
2	4	6	8	10	.	.	.
3	6	9	12	15	.	.	.
4	8	12	16	20	.	.	.
5	10	15	20	25	.	.	.

In the above array the entries in the nth row are successive multiples of n. Find the formula for S_n, the sum of the entries in the nth diagonal [e.g. S_5 = sum of the numbers in the box = 35.]

Answer: We begin by recalling the Pascal Triangle property of the binomial coefficients: $-\ {}^{k+1}C_r = {}^kC_{r-1} + {}^kC_r$. In particular, ${}^{n+1}C_2 = {}^{n+2}C_3 - {}^{n+1}C_3$ for all n.

Now subtracting from S_n the numbers in the (n-1)th diagonal yields

$$S_n - S_{n-1} = 1 + 2 + 3 + \ldots + (n-1) + n = \tfrac{1}{2}n(n+1) \text{ (Sum of A.P.)}$$

$$= {}^{n+1}C_2 = {}^{n+2}C_3 - {}^{n+1}C_3.$$

Hence if $S_{n-1} = {}^{n+1}C_3$ it follows immediately that $S_n = {}^{n+2}C_3$. Since $S_1 = 1 = {}^{1+2}C_3$ we have $S_2 = {}^{2+2}C_3$, then $S_3 = {}^{3+2}C_3, \ldots,$. In fact, by mathematical induction, for all n

$$S_n = {}^{n+2}C_3 = \frac{1}{6}n(n+1)(n+2) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$$

Q.617. Let n be a positive integer and s any real number greater than 1. Prove that

$$1/1^s + 1/2^s + 1/3^s + \ldots + 1/n^s < 1 + 1/(2^{s-1} - 1).$$

Answer: Find k such that $2^k > n$.

Then

$$\frac{1}{1^s} + \frac{1}{2^s} + \dots + \frac{1}{n^s}$$

$$\le \frac{1}{1^s} + \frac{1}{2^s} + \dots \frac{1}{n^s} + \dots + \frac{1}{(2^k - 1)^s}$$

$$< \frac{1}{1^s} + \left(\frac{1}{2^s} + \frac{1}{2^s}\right) + \left(\frac{1}{4^s} + \frac{1}{4^s} + \frac{1}{4^s} + \frac{1}{4^s}\right) + \left(\frac{1}{8^s} + \dots\right) + \dots$$

$$+ \left(\frac{1}{(2^{k-1})^s} + \dots + \frac{1}{(2^{k-1})^s}\right)$$

since we have replaced each term $\frac{1}{n^s}$ by the larger number obtained by reducing the n on the denominator to the next smaller power of 2.

The RHS of the last line

$$= 1 + 2 \times \frac{1}{2^s} + 4 \times \frac{1}{4^s} + \dots + 2^{k-1} \frac{1}{(2^{k-1})^s}$$

$$= 1 + \frac{1}{2^{s-1}} + \frac{1}{4^{s-1}} + \dots + \frac{1}{(2^{k-1})^{s-1}}$$

$$= 1 + \frac{1}{2^{s-1}} \left[\frac{1 - \left(\frac{1}{2^{s-1}}\right)^k}{1 - \frac{1}{2^{s-1}}}\right] \quad \text{(sum of a G.P. with ratio } \frac{1}{2^{s-1}} \text{.}$$

$$< 1 + \frac{1}{2^{s-1}} \frac{1}{1 - \frac{1}{2^{s-1}}} \qquad \text{Q.E.D.}$$

Q.618. Let t,u,v,w,x,y,z be non-negative numbers whose sum is 1, and let m be the maximum of $t + u + v$, $u + v + w$, $v + w + x$, $w + x + y$ and $x + y + z$. Find the smallest possible value of m, and show how to choose the numbers to achieve that value of m.

Answer: Adding the five expressions, (all $\le m$) gives

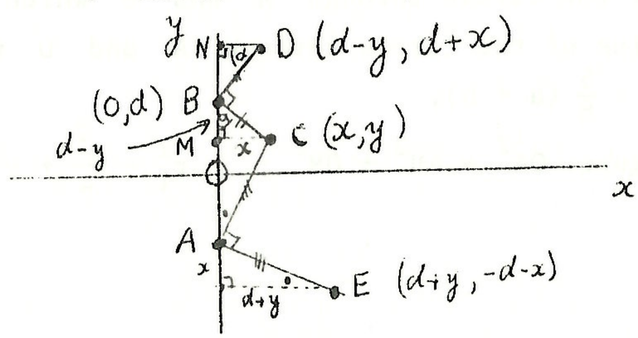
$$\text{sum} = 3(t + u + v + w + x + y + z) - t - (t+u) - (y+z) - z$$
$$\ge 3 - 4m$$

(since each of the four terms subtracted is at most equal to m). Hence $5m \ge 3 - 4m$, yielding $m \ge \frac{1}{3}$.

To achieve equality in the above we must have t, $t + u$, $y + z$ and z equal to $m = \frac{1}{3}$, as well as each of the five given expressions. Solving we easily obtain $t,u,v,w,x,y,z = \frac{1}{3},0,0,\frac{1}{3},0,0,\frac{1}{3}$.

Q.619. Two pirates bury their treasure in a field using 2 tall trees, A and B as landmarks. Not too far from the trees they find a rock, C. One pirate paces out from C to B, turns 90° right, and paces out a distance BD equal to CB. The other pirate similarly walks from C to A, turns 90° left, and goes to E, making AE equal to CA. Then they walk towards each other, meeting at the point X half way between D and E, where they bury the treasure. When they return to recover the treasure the trees are easily recognized, but owing to a landslide there are now hundreds of rocks indistinguishable from C covering the field. Can they find the treasure?

Take AB as the y-axis with the origin half way between A and B. The co-ordinates of the two trees are then (0,d) and A(0,−d) where d is half the distance between the trees. Let (x,y) be the co-ordinates of C.



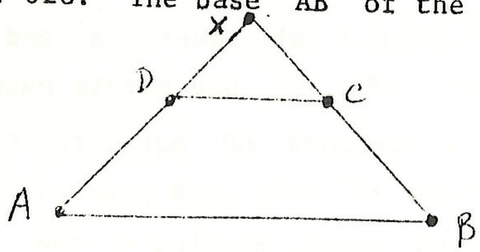
It is easy to find the co-ordinates of D viz (d−y, d+x). The figure, in which the right angled triangle BMC and DNB are easily proved congruent suggests one method.

Similarly the co-ordinates of E are found to be (d+y, −d−x). Hence the co-ordinates of X, the mid-point of DE are

$$\left(\frac{(d-y) + (d+y)}{2}, \frac{(d+x) + (-d-x)}{2} \right) = (d,0)$$

If the pirates are unable to reproduce the above mathematics, and instead just guess where C was, they will still find the treasure even if their guess was a long way off the mark.

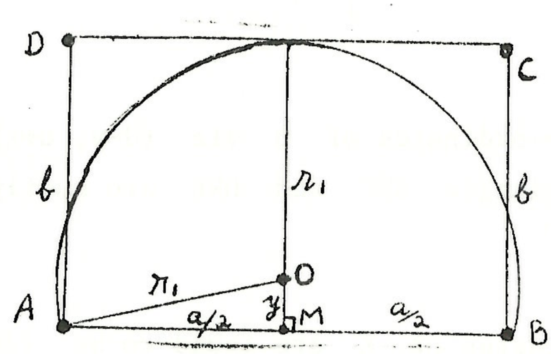
Q. 620. The base AB of the trapezium ABCD is fixed, but the parallel side CD is moved (remaining parallel to AB) so that neither its length nor the perimeter of the trapezium changed. Find the locus of the intersection X of AD and BC.



Answer: Let $k = \frac{AB}{AB-DC}$. From the similar triangles AXB and DXB we easily obtain $AX = kAD$ and $BX = k\,BC$, so that $AX + BX = k(AD + BC)$. Since $AD + BC$ remains unchanged the point X moves so that the sum of its distances from the fixed points A and B is constant. As is well known its locus is therefore an éllipse with foci at A and B.

Q.621. In the rectangle ABCD let length $AB = a$ and length $BC = b$. Let r_1 be the radius of the circle through A and B which touches CD, and let r_2 be the radius of the circle through A and D which touches BC. Prove that $r_1 + r_2 \geq \frac{5}{8}(a + b)$.

Answer: From the figure $OA^2 = AM^2 + OM^2$; $r_1^2 = \frac{a^2}{4} + y^2$.



Since $y = b - r_1$ (or $r_1 - b$ if O is outside the rectangle) we obtain $r_1^2 = \frac{a^2}{4} + (b - r_1)^2$ which yields $r_1 = \frac{b}{2} + \frac{a^2}{8b}$. Similarly $r_2 = \frac{a}{2} + \frac{b^2}{8a}$

so that $r_1 + r_2 = \frac{a+b}{2} + \frac{a^3 + b^3}{8ab} = (a+b)\left[\tfrac{1}{2} + \frac{1}{8}\,\frac{a^2 - ab + b^2}{ab}\right]$. Clearly the desired result is equivalent to $\frac{a^2 - ab + b^2}{ab} \geq 1$ which (for a,b both positive) is easily shown : $(a - b)^2 \geq 0 \Rightarrow a^2 - ab + b^2 \geq ab$

$$\Rightarrow \frac{a^2 - ab + b^2}{ab} \geq 1.$$

Q.622. Let N be any perfect square. If its digits are added together, the number $S(N)$ results. If this operation is repeated often enough, a one digit number is obtained. Show that it is always one of 1,4,7 or 9.

e.g. $3892 \times 3892 = 15147664$ and $S(S(15147664)) = 7$.

$9262 \times 9262 = 85784644$ and $S(S(S(85784644))) = 1$.

Answer: If x leaves the remainder 0,1,2,3,4,5,6,7 or 8 on division by 9, then x^2 leaves the remainder 0,1,4,0,7,7,0,4 or 1 respectively. (Check this).

Next note that $a \times 10^k - a$ is divisible by 9 for any whole numbers a and k. (Since $10^k - 1 = 999\ldots9$ (k digits)). If $N = a_0 + a_1 \times 10 + \ldots + a_n 10^n$, then $S(N) = a_0 + a_1 + \ldots. + a_n$ and $N - S(N) = a_1(10-1) + \ldots + a_n(10^n-1)$ which is divisible by 9. Thus $S(N)$ have the same remainder as N on division by 9. So do $S(S(N))$, $S(S(S(N)))\ldots$ by repetition of the argument. Since we end up eventually with a positive digit whose remainder on division by 9 is 0,1,4 or 7 that digit must be equal to one of 1,4,7 and 9.

Q.623. Andy, Bert, and Colin, having tied for first place in their chess club tournament, are to play off for the championship. Each is to play one game with both the others, scoring 1 for a win, ½ for a draw, and 0 for a loss. If their scores are still all level, this will be repeated. However if two of them are level ahead of the third, those two will continue to play until one of them scores a win.

Andy plays sound but cautious chess. He never loses against either of the others, but has probability $\frac{1}{10}$ of beating Bert in any given game, and probability $\frac{1}{5}$ of winning against Colin. Games between Bert and Colin are swashbuckling affairs that never result in draws; Bert wins 60% and Colin 40% Compare Andy's and Colin's chances of emerging as club champion.

Answer: C will win the championship only if the result of the first play off round is A v B $\frac{1}{2} - \frac{1}{2}$; A v C $\frac{1}{2} - \frac{1}{2}$; B v C: 0 - 1 which occurs with probability $.9 \times .8 \times .4 = 28.8\%$.

Similarly B will win only if the result of the final play-off round is A v B : $\frac{1}{2} - \frac{1}{2}$; A v C : $\frac{1}{2} - \frac{1}{2}$; probability $.9 \times .8 \times .6 = 43.2\%$. Therefore A's probability of winning is $(100 - 28.8 - 43.2) \doteq 28\%$.

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