A PARADOX, A PARADOX, A LOVELY LITTLE PARADOX

BY

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A QUESTION IN MECHANICS

Here is the remarkable story of Aristotle's wheel. Consider a circular wheel with a circular hubcap having the same centre as the wheel. The wheel has radius R and the hubcap has radius r. Imagine the wheel to roll along the horizontal track AB through one revolution. As this happens, the hubcap rolls along CD and also makes one revolution.

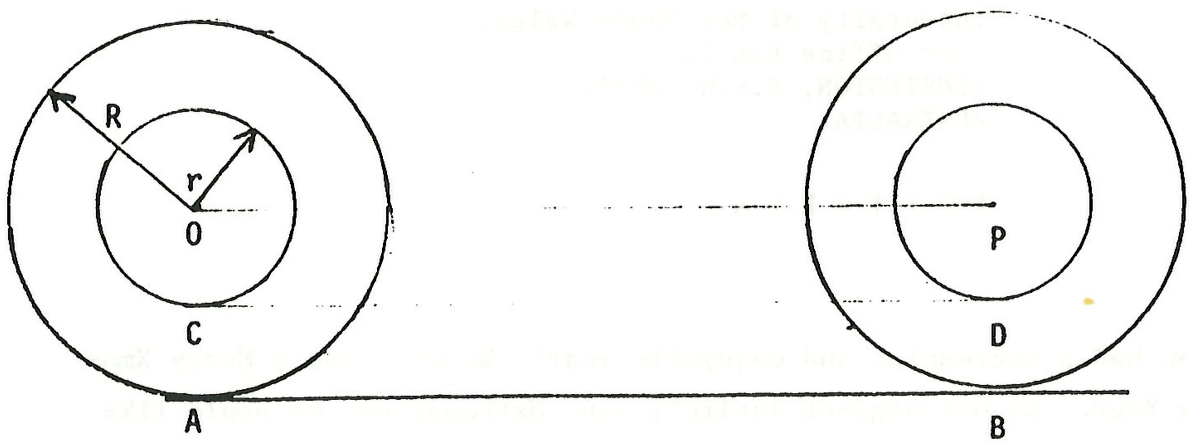


Figure 1.

The distance travelled by the wheel along AB is equal to its circumference, namely $2\pi R$. Similarly the distance travelled by the hubcap along CD is $2\pi r$. Thus $2\pi R = 2\pi r$! We have proved that any two numbers are equal. Can this be so?

Stop reading now and resolve this little paradox for yourself. It is still not too late to stop. In any case, there is absolutely nothing of consequence in the rest of this paragraph, so you might just as well stop reading before it is too late. If all numbers are equal, then 0 = 1, which shows that it is possible to create something out of nothing. This is a proof that god exists. You should be able to do better than this. By now it must be clear that you should have stopped reading when you were told. This is not a sentence. If you have got this far, you now have another paradox to explain, so stop now before something even worse happens.

Aristotle lived from 384 to 322 B.C. and founded the Lyceum, the successor to Plato's Academy, in Athens in 335 BC. He was one of the great philosophers of all time and wrote on mechanics, physics, mathematics, logic, meteorology, botany, psychology, zoology, ethics, literature, metaphysics, economics and many other things. His influence controlled the direction of scientific thought at least until the sixteenth century and there are those who would say that this was not a good thing. Aristotle believed in the power of the mind to reveal the causes for things. Sometimes, pure thought can be misled.

Aristotle maintained that the natural place for a heavy body is the centre of the universe which is the centre of the earth; that is why a heavy body falls. The velocity, V, of a falling body increases with the force, F, provided by its weight, and is decreased by the resistance, R, provided by the medium. We might write $V = cF/R$. Heavier bodies fall faster than lighter ones because the force provided by the weight is larger. In a vacuum, there would be no resistance, so the velocity would be infinite. Thus a vacuum is impossible. This does not account for the fact that the velocity increases as the body falls. Aristotle said that this happens because the body moves more jubilantly as it nears its natural place.

A DISCOURSE ON SCIENCE

One of founders of modern science was Galileo (1564-1642). For Galileo, first principles came from experience and experimentation, rather than from the power of the mind. (Of course, if the result of an experiment was obvious, it was not necessary to go to the trouble of performing it.) So Galileo observed falling bodies and saw that he should start by finding the laws of motion in a vacauum and only try to account for the difficulties of friction and resistance after that. He discovered that the velocity, V, of a body falling in a vacuum is 32T feet persecond, T seconds after it starts falling from rest. Thus he obtained quantitative laws inaccessible to Aristotle's methodology. He replaced Aristotle's complicated explanations and causes by simple descriptions and this is more or less how science operates today. The correct theory is just complicated enough to explain the observations, but not more so. Naturally, Galileo was forever in strife with the church which found much comfort in the dogmas and mysticism of Aristotle.

Galileo's "Dialogues concerning two new sciences"(1638) contains a remarkable analysis of Aristotle's wheel. If you have found the obvious resolution of the paradox, this may explain why the wheel is a paradox after all.

To begin, consider two concentric hexagons $A_1A_2 \dots A_6$ and $C_1C_2 \dots C_6$. Imagine the larger hexagon to roll along A_1B carrying the smaller hexagon with it. If the

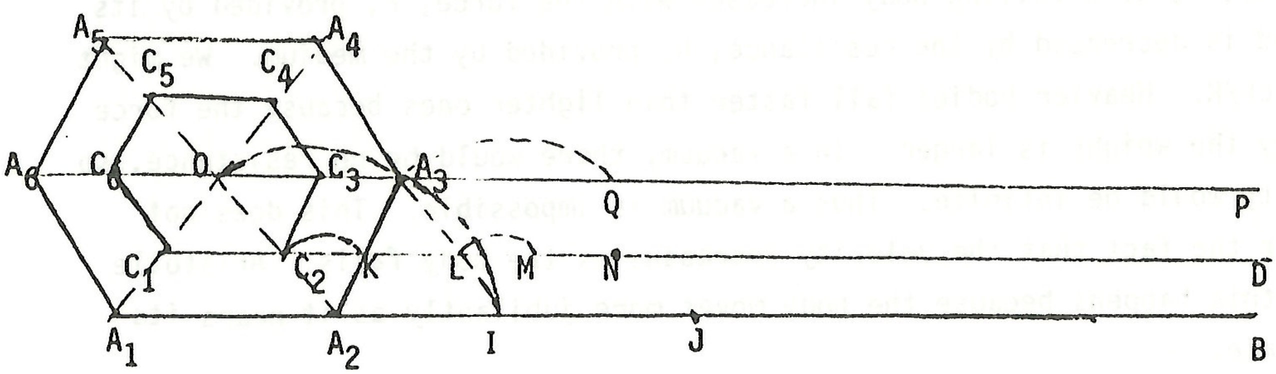


Figure 2.

point A_2 remains fixed the point A_1 will rise and the points A_3 will fall describing the are A_3I until the side A_2A_3 concides with the line A_2I. During this rotation, the point C_2 on the smaller hexagon will follow the arc C_2K and the side C_2C_3 will reach the segment KL. The centre O will follow the arc OA_3. This brings the figure to a position similar to its starting position.Continue the rotation with centre I. The side A_3A_4 of the larger hexagon will reach IJ, the side C_3C_4 of the smaller hexagon will reach MN after skipping over the arc LM, and the centre will reach Q after jumping over A_3Q. After one complete revolution, the larger hexagon will have traced six segments along the line A_1B whose total length is its circumference. The smaller hexagon will have imprinted six segments in total equal to its circumference along the line C_1D, but separated by five arcs whose chords are the parts of C_1D not touched by the hexagon. The centre touches the line OP at just six points. The distance travelled by the smaller hexagon is nearly equal to that travelled by the larger, provided we understand the line C_1D to include the five skipped arcs.

The same analysis applies if the hexagons are replaced by polygons with, say, 1000 sides. In one complete revolution, the larger polygon will lay off a line equal to its perimeter, and the smaller one will pass over an approximately equal distance made up of a thousand small segments separated by a thousand empty segments which the polygon has skipped.

Now, what happens when the polygons have infinitely many sides and become circles, as in Aristotle's wheel? In one revolution, the larger circle rolls along the line AB equal to its circumference. (See figure 1.) The centre O moves the same distance along the line OP. The smaller circle touches every point along the equal line CD without skipping any vacant spaces. How can the smaller circle traverse a length greater than its circumference unless it goes in jumps?

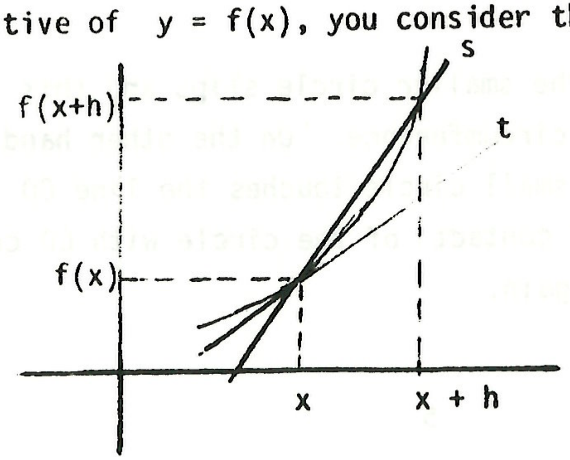
PARADOX REGAINED

Any clown can see that the smaller circle slips and this is how it travels a distance greater than its circumference. On the other hand, it is still a little strange. No point on the small circle touches the line CD in more than one point because the point of contact of the circle with CD changes continuously. Things are slipping away again.

Galileo argues the position as follows. In the case of polygons with 100000 sides, the smaller polygon traces out a line comprising 100000 segments interspersed with 100000 empty spaces. In the case of circles, that is polygons with infinitely many sides, the smaller circle traces out a line comprising infinitely many segments interspersed with infinitely many empty spaces. The line traversed by the larger circle consists of infinitely many points which completely fill it. The line traversed by the smaller circle consists of an infinite number of points which leave empty spaces and only partly fill the line. These points amount to a total length of $2\pi r$ and the spaces fill out the rest of the total length of CD which is $2\pi R$. It seems that Galileo's argument is in imminent danger of disappearing through the holes in his lines.

In essence, the problem is to explain how a continuous line can be built up from an infinite number of indivisible points. If a point has zero length, then how can any number of points add up to a positive length. If a point has a positive length, then infinitely many points will add up to an infinite length. In one form or another, this problem has bedevilled mathematicians since the time of the Ancient Greeks. Zeno in about 450 B.C. illustrated it with the paradox of Achilles and the Tortoise. Suppose the tortoise starts at T and Achilles starts at A and they both run like mad to the right starting now. When Achilles reaches

T, the tortoise will have moved to O; when Achilles reaches O the tortoise will have moved to R, and so on. Achilles will never catch the tortoise. The discoverers of the calculus in the seventeenth century met similar difficulties. To define the derivative of $y = f(x)$, you consider the ratio



$$\frac{f(x + h) - f(x)}{h} ,$$

where h is an infinitesimal increment. Now, if h = 0, this ratio is o/o which is meaningless; if h > 0, then the ratio gives the slope of the secant s and not the slope of the tangent t. It took many centuries to resolve these difficulties and there is fortunately no space to pursue the story here. There are some hints in "The Mathematical Experience" by P.J. Davis and R. Hersch (Harvester, 1981), particularly chapters 4 and 5, and in "Riddles in Mathematics" by E.P. Northrop (Pelican, 1963), Chapter 7.

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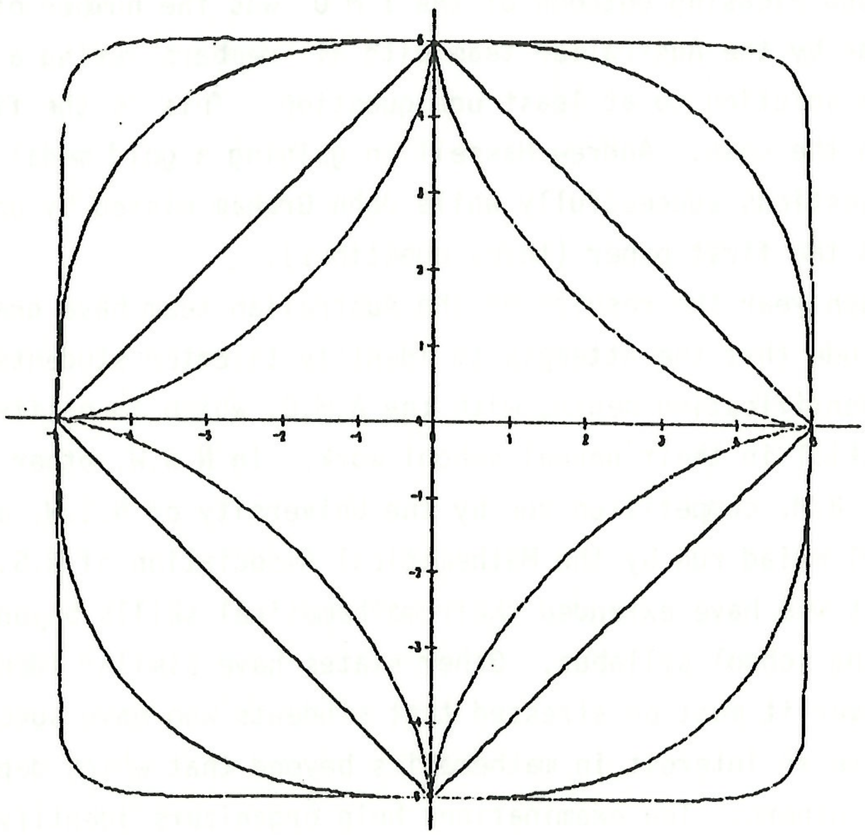

$$\left(\frac{x}{5}\right)^n + \left(\frac{y}{5}\right)^n = 1, \qquad 0.7 \leq n \leq 10.$$