LEARN TO FIND YOUR WAY IN THREE-DIMENSIONS!

We live and function in a three-dimensional space all through our existence, we are surrounded almost exclusively by three dimensional objects, yet when we have to describe spacial processes, geometrical objects, and their relationships by abstract presentations in the plane: we run into difficulties. Here we look at some examples which will show how to tackle such problems.

We are all familiar with the parabola. If we throw a ball up at an angle its path is a pretty good approximation to the parabola. This path lies in a plane which is perpendicular to the surface of the earth. We can define this curve as the locus of a point which is equidistant from a fixed point, F, and from a fixed line, ℓ, where F does not lie in ℓ. We call the fixed point, F, the focus and the fixed line ℓ the directrix of the parabola. We can conveniently illustrate our curve in the Cartesian co-ordinate system as shown in Figure 1. In this situation we have the equation

$$y = x^2 \qquad (1)$$

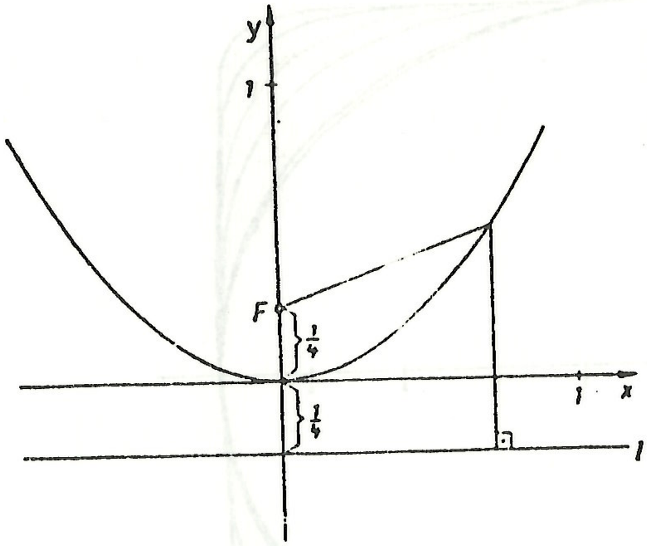


Figure 1

representing the parabola. With our definition we managed to represent a one-dimensional point set (or a one-dimensional geometrical object, a curve) in the two dimensional plane. The easist step from two- into three-dimensional space is through the introduction of rotation. If we rotate our parabola about its axis (in our present example it coincides with the y-axis), the parabola sweeps out a surface which we call a paraboloid of rotation. This surface inherits the properties of the generating parabola. For example F, the focal point, remains the focus of the paraboloid, so if a light source is placed at F the reflected rays from the "inner" surface travel parallel to the axis of the parabola. We know that the properties of the paraboloid of rotation have wide and important applications in present day technology and science.

Next we shall look at a more difficult example in three dimensional space. Let g and h be two striaght lines which are perpendicular to each other without intersecting. This, of course, is impossible in two-dimension, but perfectly all right in three-dimension. Think of a bridge spanning a roadway. Two such straight lines are called "skew", or "crossing" lines when they have no common point, neither are they parallel to each other. The "crossing angle" of two skew straight lines is equal to the angle formed by two intersecting straight lines which are parallel to the respective skew lines. So if two skew straight lines are perpendicular to each other, the corresponding parallel straight lines intersect each other at right angles. Note that we talk about "crossing" and "intersecting" straight lines to distinguish between skew lines and lines lying in the same plane.

We now ask the question: what is the geometrical object formed by all those points which are equidistant from two perpendicularly "crossing", skew straight lines g and h?

Note: The distance of a point P from a straight line g is the length from P to the foot of the perpendicular from P to g. (See Figure 2.) Let us now analyse our problem. It is clear that the set of points satisfying the stated condition will form a surface and this surface shares no common points with g and h. Furthermore a plane α containing h and perpendicular to g will cut this surface in a parabola, where h is the directrix and the point Q where g pierces the plane α is the focus of this parabola. Similarly a plane, β, through g and perpendicular to h will cut the surface in a parabola.

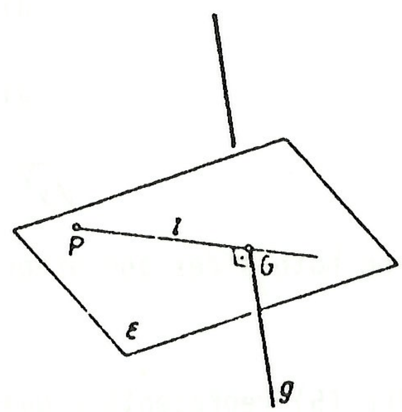


Figure 2

We can illustrate this three-dimensional situation on our two-dimensional paper, keeping in mind that the x-axis should project out of the plane. We take g, one of our straight lines parallel to the x-axis, so that it passes through the point $Q(0,0,a)$, take the other straight line h parallel to the y-axis, so that it passes through the point $R(0,0,-a)$. The lines g and h "cross" perpendicular at a distance $2a$. (See Figure 3.)

Let $P(x,y,z)$ be the general point satisfying the required condition. The foot of the perpendicular from P to g is $G(x,0,a)$, the foot of the perpendicular from P to h is $H(0,y,-a)$. According to our requirement

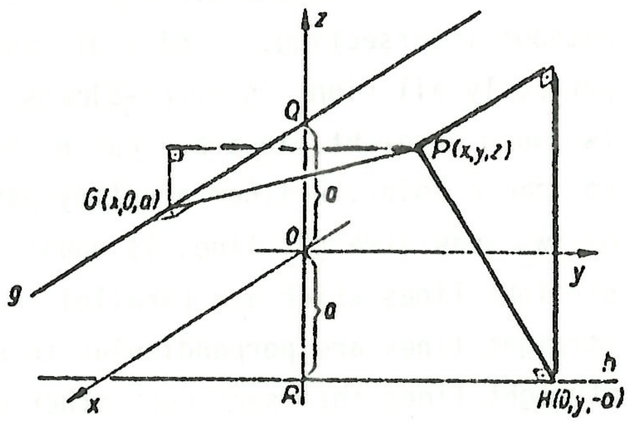


Figure 3

$$\text{distance (PG)} = \text{distance (PH)} \tag{2}$$

Using the distance formula (or Pythagoras's Theorem) we have

$$\text{dist (PG)} = \sqrt{y^2 + (z-a)^2}$$

$$\text{dist (PH)} = \sqrt{x^2 + (z+a)^2} \tag{3}$$

and

$$\sqrt{x^2 + (z+a)^2} = \sqrt{y^2 + (z-a)^2} \tag{4}$$

Squaring both sides and ordering the equation we get

$$4az = y^2 - x^2 \tag{5}$$

Equation (5) represents a quadratic surface Φ. To visualise and to analyse Φ we shall cut it with various planes.

1. Cut Φ with the yz-plane, i.e. put $x = 0$ into equation (5), we get

$$4az = y^2 \tag{6}$$

Comparing (6) with (1) we recognize that the curve of intersection is a parabola "open above".

2. Cut Φ with the xy-plane, i.e. put $y = 0$ into equation (5), we get

$$4az = - x^2 \tag{7}$$

The curve of intersection is clearly a parabola "open downwards".

3. Cut Φ with the xy-plane, i.e. put $z = 0$ into equation (5), we get

$$y^2 - x^2 = 0 \tag{8}$$

which can be written as

$$(y - x)(y + x) = 0 \tag{9}$$

A product vanishes when one of its factor becomes zero. So from (9) we obtain the equations

$$y = x \quad \text{and} \quad y = - x \tag{10}$$

So the curve of intersection splits into the two straight lines represented by the equations (10).

4. Cut Φ with a plane parallel to the yz-plane. We put $x = k$ into equation (5) and obtain

$$4az = y^2 - k^2 \tag{11}$$

Clearly the curve of intersection is a parabola "open above". Comparing (11) and (6) we see that with shifting a parallel plane, we are obtaining parabolas as curves of intersections.

5. Cut Φ with a plane parallel to the xy-plane. We put $y = k$ into equation (5) and obtain

$$4az = k^2 - x^2 \tag{12}$$

The curve of intersection is a parabola "open downwards", and so are all the others cut by these family of planes.

We observe that there are two families of parabolas lying on the surface Φ. We may generate the surface Φ by parallel shifting the parabola lying in the xy-plane along the parabola lying in the yz-plane, so that the vertex of the first parabola lies always on the second parabola. (see Figure 4.)

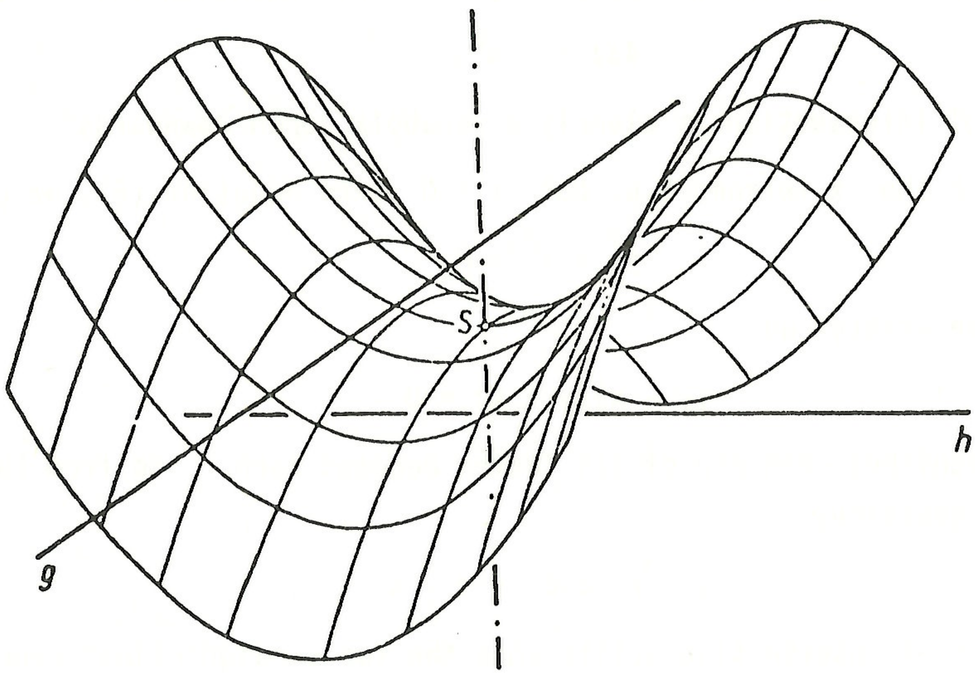


Figure 4

6. To have a deeper understanding of our surface cut Φ with a plane which is parallel to the xy-plane. Putting $z = k$ into (5) we obtain

$$y^2 - x^2 = 4ak \tag{13}$$

For $k = \pm 1, \pm 2, \pm 3, \ldots$ we obtain a family of equilateral hyperbolas. If we project these hyperbolas onto the xy-plane we get the "strata-plan" illustrated in Figure 5. This strata-plan includes the special case of $y^2 - x^2 = 0$, resulting in the two straight lines obtained earlier. These lines are assymptotes to the family of hyperbolas. This surface Φ - for obvious reasons - is frequently called the saddle-surface. But considering the

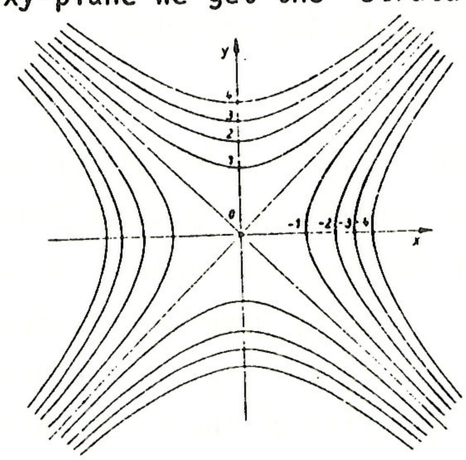


Figure 5

resulting curves of intersections we obtained by various cutting planes, Φ's geometrical name is the hyperbolic paraboloid.

We must consider one more important series of cuts.

7. Cut Φ with a plane parallel to the plane $y = x$. Putting $y = x + k$ into equation (5) we obtain

$$2kx + k^2 = 4az \tag{14}$$

Here we have a linear equation in x and z which indicates that these cuts with the family of planes $y = x + k$ result in a family of straight lines. The same applies to cuts obtained by the family of planes $y = -x + k$.

So we have discovered the interesting fact that there are two distinct families of straight lines which can be laid onto the curved surface Φ! This is certainly not obvious at a glance! This important property of Φ makes it possible to be constructed from straight line elements, hence its wide application in architecture, building and industry. Figure 6 shows how this "saddle-surface" arises as the diagonally opposite corners of a square rid are raised and lowered respectively.

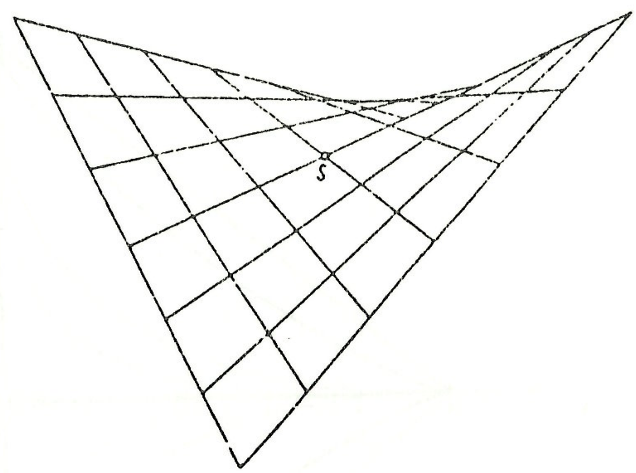


Figure 6

To encourage you to exercise your spacial imagination and your analytic power I leave you with an assignment:

Find the geometrical object formed by all the points which have the distance - relationship from two perpendicular skew lines g and h expressed as

(continued over)

$$\frac{\text{distance(GP)}}{\text{distance(HP)}} = \frac{1}{K}$$

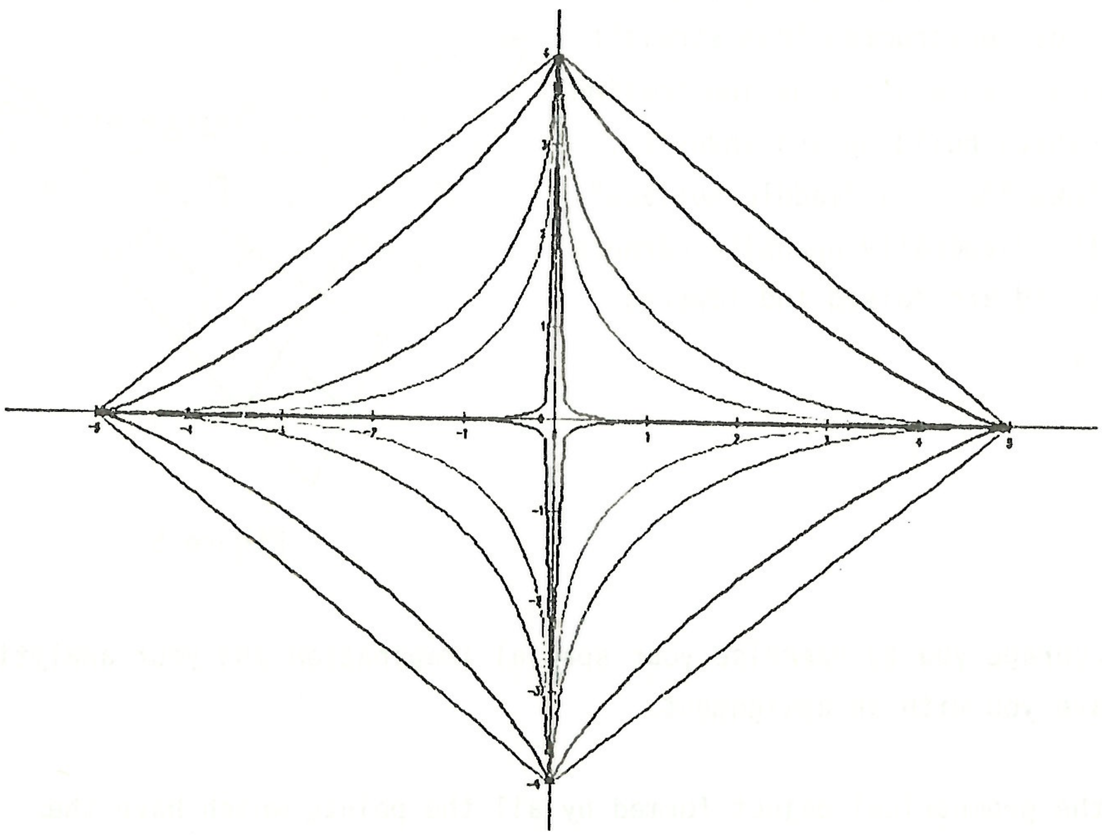
where $K > 1$.

You can send your work by no later than 31st January 1986 to me

Editor of Parabola.

I have a special book prize for best work!

This article was based on the work: "Von der zweiten in die dritte Dimension" by E. Schröder (ALPHA/15/1981).



$$\left(\frac{x}{5}\right)^n + \left(\frac{y}{4}\right)^n = 1, \quad 0.2 \leqslant n \leqslant 1.00.$$