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THE WORLD OF MATHEMATICS
IS SET-THEORY ANY USE?

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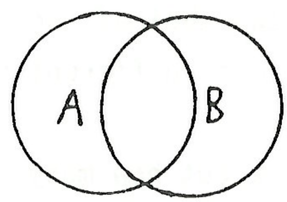
Everybody learns some set theory early in high school. Most people find it easy to pick up the ideas - subsets, unions, intersections. (There have even been kindergarten textbooks on set theory). Then the subject disappears from the course. Was there any point in it? Was it all as trivial as it looked? Are there any hard results in the subject? Or was it all just a trendy fad?

Well, there is at least one kind of problem that set theory casts some light on: If 55 people shop in a complex containing shops A and B, and 27 shop at A and 38 at B, how many shop at both? The answer is 38 + 27 - 55 (=10), as is clear from the diagram of sets:



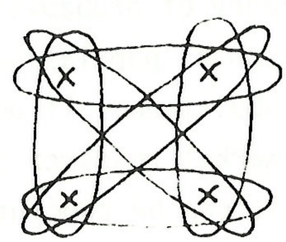
But that still doesn't give set theory much work to do. Is there anything else?

In fact there is, in the theory of probability and statistics. Opinion polls involve taking a "sample" from a "population" and then supposing that the sample is representative of the population. If people can prefer either floral or plain tops this spring, and a market research survey finds that 70% of a large sample prefer floral, then it can be concluded with reasonable certainty that about 70% of the population prefer floral. Surveys work because of the purely mathematical fact that the great majority of large samples resemble the population in composition. Now a "population" is just a set, and a "sample" is just a subset. So the mathematical question is one in set theory: given a set consisting of two kinds of things, what proportion of subsets of a given size have a composition close to that of the population? An example of the answers found is:

Most samples of size 100 (from any population of two kinds) have composition within 4% of the population composition.

So, set theory helps you count (and, in fact, there is not much you can do with sets except count them, or count the things in them). The obvious first problem then is: how many subsets does a set have? It is clear that the answer depends only on the <u>size</u> of the set - what the names of the objects are makes no difference. What about a set of size 3, {a,b,c}? The subsets are (counting the empty set and the whole set, as is conventional): {}, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}. There are 8. The simplest connection between 3 and 8 is that $8 = 2^3$. We might conjecture that the number of subsets of a set of size n is 2^n. Calculating with a set of 4 elements {a,b,c,d} would confirm this - there are 16 subsets. 2^n is the right answer, but on the face of it it is hard to explain why. If the question is just about sets, subsets and the number n, how can 2 come into the picture? The answer is this: we decide on a subset by deciding, for each of the n elements, whether it will be in the subset or not. So for each element, there are two choices: in the subset or out. So there are altogether 2 x 2 x ... x 2 (n times) different subsets, that is, 2^n.

The next problem is to work out how many subsets of a set there are of a given size. For example, a set of 4 things has 6 subsets of size 2:



There is much that can be said about the number of r-element subsets of a set of size n (usually written ${}^nC_r$). But not now.