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This issue we first look at the problems on inequalities:

Question 85.3 (3 unit paper)

It is given that $A > 0$, $B > 0$ and n is a positive integer.

(a) Divide $A^{n+1} - A^nB + B^{n+1} - B^nA$ by $A - B$, and deduce that

$$A^{n+1} + B^{n+1} \geqslant A^nB + B^nA.$$

(b) Using (a), show by mathematical induction that

$$\left\{\frac{A + B}{2}\right\}^n \leqslant \frac{A^n + B^n}{2}.$$

Solution: (a) Clearly $A^{n+1} - A^nB + B^{n+1} - B^nA$.

$$= A^n(A - B) + B^n(A - B),$$

$$= (A^n - B^n)(A - B), \qquad (1)$$

By inspection for the 3 cases $A > B$, $A = B$, $A < B$, the expression in (1) $\geqslant 0$. [or $(A - B)(A^n - B^n) = (A - B)^2(A^{n-1} + A^{n-2}B + \ldots + B^{n-1}) \geqslant 0$].

Thus

$$A^{n+1} + B^{n+1} \geqslant A^nB + B^nA.$$

(b) Assume $\left(\frac{A + B}{2}\right)^k \leqslant \frac{A^k + B^k}{2}$, where k is a positive integer.

Thus $$\left(\frac{A + B}{2}\right)^{k+1} \leqslant \left(\frac{A + B}{2}\right)\left(\frac{A^k + B^k}{2}\right) = \tfrac{1}{4}[A^{k+1} + B^{k+1} + BA^k + AB^k].$$

But, by (a) $BA^k + AB^k \leqslant A^{k+1} + B^{k+1}$, therefore

$$\tfrac{1}{4}[A^{k+1} + B^{k+1} + BA^k + AB^k] \leqslant \tfrac{1}{4}[A^{k+1} + B^{k+1} + A^{k+1} + B^{k+1}],$$

$$\leqslant \tfrac{1}{2}(A^{k+1} + B^{k+1}).$$

Thus, if the result is true for $n = k$, then it is also true for $n = k + 1$.

But, for $n = 1$,

$$\frac{A + B}{2} \leqslant \frac{A + B}{2},$$

therefore result is true for $n = 1$, and hence it is true for all positive integers n.

Question 85.4 (4 unit)

It is given that x,y,z are positive numbers. Prove that

(a) $x^2 + y^2 \geqslant 2xy$,

(b) $x^2 + y^2 + z^2 - xy - yz - zx \geqslant 0$.

Multiply both sides of the inequality (b) by $(x + y + z)$ to obtain

(c) $x^3 + y^3 + z^3 \geqslant 3xyz$.

Deduce from (c), or prove otherwise, that

(d) $(x + y + z)(x^{-1} + y^{-1} + z^{-1}) \geqslant 9$.

Suppose that x,y,z satisfy the additional constraint that

$$x + y + z = 1.$$

Is it true that the minimum value of the expression

$$x^{-1} + y^{-1} + z^{-1}$$

is equal to 9? Justify your answer.

Solution: (a) This is a gift, of course. In case you have forgotten

$$x^2 + y^2 = (x - y)^2 + 2xy \geqslant 2xy.$$

(b) Another well-known question. Consider

$$X = (x - y)^2 + (y - z)^2 + (z - x)^2 \geqslant 0.$$

Expanding

$$X = x^2 + y^2 - 2xy + y^2 + z^2 - 2yz + z^2 + x^2 - 2xz$$

$$= 2[x^2 + y^2 + z^2 - xy - yz - zx] \geqslant 0.$$

(c) Since $x + y + z > 0$,

$$0 \leqslant (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx),$$
$$= x^3 + y^3 + z^3 + xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2$$
$$- x^2y - xyz - x^2z - xy^2 - y^2z - xyz - xyz - yz^2 - z^2x,$$
$$= x^3 + y^3 + z^3 - 3xyz.$$

(d) This is a very pretty consequence of (c) - and easy, provided you are not put off by the notation. Better still, replace x,y,z by a,b,c. Then, if $a = x^3$, $b = y^3$, $c = z^3$, then $a + b + c = x^3 + y^3 + z^3 \geqslant 3xyz = 3a^{1/3}b^{1/3}c^{1/3}$. Similarly

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geqslant 3a^{-1/3}b^{-1/3}c^{-1/3}.$$

Thus for positive a,b,c,

$$(a + b + c)(a^{-1} + b^{-1} + c^{-1}) \geqslant 3a^{1/3}b^{1/3}c^{1/3} . 3a^{-1/3}b^{-1/3}c^{-1/3} = 9.$$

The remainder of the question is a little misleading. Simply take $a = b = c = t$ (say), then

$$a + b + c = 3t$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{3}{t},$$

and then

$$(a + b + c)(a^{-1} + b^{-1} + c^{-1}) = 9.$$

Thus the minimum of the stated expression is 9. If $a + b + c = 1$, then $a = b = c$ implies that $a = b = c = \frac{1}{3}$, and then the expression is actually 9.

We now look at some of the trigonometric problems:

Question 85.5 (3 unit)

Prove that: $8 \cos^4 x = 3 + 4 \cos 2x + \cos 4x$.

Solution: Note that $2 \cos^2 x = 1 + \cos 2x$. Therefore

$$8 \cos^4 x \equiv 2(2 \cos^2 x)^2 \equiv 2(1 + \cos 2x)^2$$
$$\equiv 2 + 4 \cos 2x + 2 \cos^2 2x$$
$$\equiv 2 + 4 \cos 2x + 1 + \cos 4x$$
$$\equiv 3 + 4 \cos 2x + \cos 4x.$$

Question 85.6 (4 unit)

Write down expressions for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$ in terms of $\sin \alpha$, $\cos \alpha$, $\sin \beta$, $\cos \beta$. Deduce that

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta},$$

and

$$\tan(\alpha + \beta + \gamma) = \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \alpha \tan \gamma}. \qquad (1)$$

By means of the substitution $t = \tan \theta$, transform the equation

$$\sin 4\theta + a \sin 2\theta + b \cos 2\theta + b = 0$$

into a cubic equation in t. (a, b are real constants, $a \neq 2$.) Suppose the roots of the transformed equation are $\tan \alpha$, $\tan \beta$, $\tan \gamma$. Show that $\alpha + \beta + \gamma$ is a multiple of π.

Solution: The first two results are quite straight-forward. Assuming these results, let $t = \tan \theta$, then:

$$\cos 2\theta = \frac{1 - t^2}{1 + t^2}, \quad \sin 2\theta = \frac{2t}{1 + t^2},$$

and

$$\sin 4\theta = 2 \sin 2\theta \cos 2\theta = \frac{2.2t(1 - t^2)}{(1 + t^2)^2}.$$

Thus

$$Z = \sin 4\theta + a \sin 2\theta + b \cos 2\theta + b,$$
$$= \frac{4t(1 - t)^2}{(1 + t^2)^2} + \frac{2at}{1 + t^2} + \frac{b(1 - t^2)}{1 + t^2} + b.$$

Thus

$$(1 + t^2)^2 Z = 4t(1 - t^2) + 2at(1 + t^2) + b(1 + t^2)(1 - t^2) + b(1 + t^2)^2$$
$$= 4t - 4t^3 + 2at + 2at^3 + b - bt^4 + b + 2bt^2 + bt^4$$
$$= t^3(2a - 4) + 2bt^2 + (4 + 2a)t + 2b.$$

Divide by two, and noting that $Z = 0$, the required cubic equation is

$$(a - 2)t^2 + bt^2 + (a + 2)t + b = 0. \qquad (2)$$

Let $t_1 = \tan\alpha$, $t_2 = \tan\beta$, $t_3 = \tan\gamma$ be the three roots of (2), then, since $a \neq 2$,

$$t_1 + t_2 + t_3 = -\frac{b}{a - 2}, \quad t_1t_2 + t_2t_3 + t_3t_1 = \frac{a + 2}{a - 2}, \quad t_1t_2t_3 = -\frac{b}{a - 2}.$$

Hence, using (1),

$$\tan(\alpha + \beta + \gamma) = \frac{t_1 + t_2 + t_3 - t_1t_2t_3}{1 - (t_1t_2 + t_2t_3 + t_3t_1)} = \frac{-\frac{b}{a - 2} + \frac{b}{a - 2}}{1 - \frac{a + 2}{a - 2}} = 0.$$

Since $\tan(\alpha + \beta + \gamma) = 0$, it follows that

$$\alpha + \beta + \gamma = n\pi.$$