SOLUTIONS TO PROBLEMS FROM VOLUME 21, NUMBER 1

Q. 624. Three motorists A, B, and C often travel on a certain highway, and each motorist always travels at a constant speed. A is the fastest of the three, and C the slowest.

One day B overtakes C, five minutes later A overtakes C and in another 3 minutes A overtakes B. The next day, A overtakes B first, then, nine minutes later, overtakes C. When will B overtake C?

Solution: Let the speeds of the motorists be v_A, v_B, v_C km/minute respectively. In 8 minutes after B overtakes C he gets a distance $8(v_B - v_C)$km ahead of C. In 3 minutes after A overtakes C he gets a distance of $3(v_A - v_C)$km ahead of C. Since at the instant when A overtakes B on the first day they are both the same distance ahead of C, we deduce

$$8(v_B - v_C) = 3(v_A - v_C). \qquad (1)$$

If on the second day B overtakes C x minutes after A overtakes C, since they are at that instant the same distance behind A,

$$(9 + x)(v_A - v_B) = x(v_A - v_C)$$

$$9v_A + xv_C = (9 + x)\, v_B.$$

From (1)

$$9v_A + 15v_C = 24\, v_B.$$

Subtracting

$$(15 - x)\, v_C = (15 - x)\, v_B$$

therefore

$$x = 15 \quad \text{since} \quad v_B \neq v_C.$$

Q. 625. At a party, each boy shakes hands with an odd number of girls, and each girl shakes hands with an odd number of boys. Show that the total number of boys and girls is even.

Solution: Add up the numbers of boy-girl handshakes made by every person at the party. If there are n people present, this is the sum of n odd numbers, which is odd if n is odd and even if n is even. But since every such handshake has been counted twice (once for each of the two who shook hands) we must end up with an even total. Thus n must be even.

Q. 626. Five coins appear to be identical, but two of them are counterfeits. One is lighter and one is heavier than a good coin, but together they exactly counterbalance two good coins. Show how to identify all the coins in three weighings, using a beam balance.

Solution: Label the coins 1, 2, 3, 4, 5.

First weighing 1 v 2.

Second weighing 1 + 2 v 3 + 4.

Case 1. 1 = 2.

Then 1 & 2 are known to be both good. If 1 + 2 = 3 + 4 then 3 & 4 are the two false coins and the 3rd weighing 3 v 4 would identify which was which. If 3 + 4 is heavier (lighter) than 1 + 2 then 5 is the light (heavy) coin and 3 v 4 identifies the heavy (light) coin in the third weighing.

Case 2. 1 > 2.

Then either (a) 1 is heavy and 2 is good

or (b) 1 is good and 2 is light

or (c) 1 is heavy and 2 is light.

If 1 + 2 = 3 + 4, (c) applies, and no further weighing is necessary.

If 1 + 2 > 3 + 4, (a) applies and the third weighing 3 v 4 will enable 3, 4 or 5 to be identified as the light coin.

If 1 + 2 < 3 + 4, (b) applies, and the third weighing 3 v 4 will enable 3, 4 or 5 to be identified as the heavy coin.

Case 3. 1 < 2.

The discussion is identical with that in Case 2, with the words heavy and light interchanged.

Q. 627. The probability that any letter reaches its destination is 4/5. I post a letter to a friend. If he received it he would certainly have sent a reply, but I received no reply. What is the probability that he received my letter?

Solution: Suppose I send a large number N letters to my firend. He receives and answers (about) $\frac{4}{5}N$, and I receive back (about) $\frac{16}{25}N$ letters. Of the $\frac{9}{25}N$ occasions in which no return letter is received, there were $\frac{1}{5}N$ in which my friend did not receive my letter, and $\frac{4}{25}N$ in which the return letter went astray. Therefore the probability that he received my letter is

$$\frac{\frac{4}{25}N}{\frac{9}{25}N} = \frac{4}{9}.$$

Q. 628. Prove that if (x_1,y_1); (x_2,y_2); and (x_3,y_3) are three points in the Cartesian plane which are vertices of an equilateral triangle, it is impossible that all the co-ordinates are integers.

Solution: Suppose it is possible. Since the origin can be translated to any vertex of the triangle, one may assume without loss of generality that the vertices of the triangle in anticlockwise order are $(0,0)$; (x_1,y_1); and (x_2,y_2) where x_1, y_1, x_2, y_2 are integers, $y_2 \neq 0$.

We have $x_1^2 + y_1^2 = x_2^2 + y_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2$ where d is the side length of the equilateral triangle. One deduces easily that

$$x_1x_2 + y_1y_2 = \tfrac{1}{2}d^2 \qquad (1)$$

The area of an equilateral triangle is $\frac{1}{2}d^2 \frac{\sqrt{3}}{2}$. Hence

$$x_1y_2 - x_2y_1 = d^2 \frac{\sqrt{3}}{2}. \qquad (2)$$

Adding x_2 times (1) to y_2 times (2) gives

$$x_1(x_2^2 + y_2^2) = \left(\frac{x_2}{2} + \frac{y_2\sqrt{3}}{2}\right)d^2$$

whence

$$x_1 = \frac{x_2}{2} + \frac{y_2\sqrt{3}}{2}$$

$$\sqrt{3} = \frac{2}{y_2}\left(x_1 - \frac{x_2}{2}\right).$$

But this is impossible since $\sqrt{3}$ is irrational, but the RHS is clearly rational. Hence it is impossible that all co-ordinates of the vertices should be integers.

Q. 629. A "uni-digit number" is one whose ordinary decimal expression consists of only one digit repeated a number of times; e.g., 44 or 7777 or 9,999,999. Prove that a uni-digit number greater than 10 is not a perfect square.

Solution: If the last digit of x is 0, 1, 2, ... 9 then the last digit of x^2 is 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 respectively; i.e. the only non-zero digits which can be the final digits of a perfect square are 1, 4, 5, 6 & 9. Now 5 ... 55 is a multiple of 5, but not of 5^2, so is not a perfect square. Similarly 6 ... 66 is a multiple of 2, but not of 2^2. Note that

$$4 \ldots 44 \; (= 2^2 \times 1 \ldots 11)$$

and

$$9 \ldots 99 \; (= 3^2 \times 1 \ldots 11)$$

are perfect squares only if 1 ... 11 is a perfect square. Since x^2 ends with 1 only if x ends in 1 or 9, we need consider only $(10a \pm 1)^2 = 1 \ldots 11$. But $(10a \pm 1)^2 = 20(5a^2 \pm a) + 1$ exceeds by 1 a multiple of 20; hence its second last digit must be even. Therefore 1 ... 11 is never a perfect square, and the proof is complete.

Q. 630. $n!$ (read n factorial) is defined to be the product of all the integers from 1 to n inclusive; i.e.

$$n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n.$$

(a) Which is larger, $\sqrt[8]{(8!)}$ or $\sqrt[9]{(9!)}$?

(b) Simplify $1(1!) + 2(2!) + 3(3!) + \ldots + n(n!)$.

Solution: (a) Obviously $8! = 1.\ 2.\ \ldots\ 8 < 9^8$. Multiplying through by $(8!)^8$ gives

$$(8!)^9 < (9!)^8$$

therefore

$$(8!)^{9/72} < (9!)^{8/72}$$

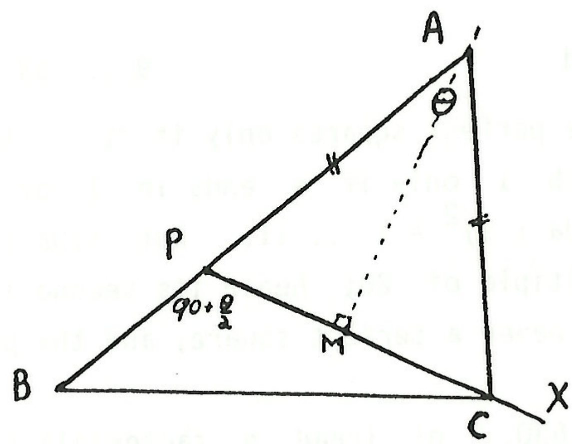
$$\sqrt[8]{8!} < \sqrt[9]{9!}$$

(b) Let $x_n = 1(1!) + 2(2!) + \ldots + n(n!)$. Then $x_1 = 1$, $x_2 = 5$, $x_3 = 23$, $x_4 = 119$, ... One guesses that $x_n = (n+1)! - 1$, and proves it by mathematical induction. Assuming that this result is true for some n consider

$$x_{n+1} = 1(1!) + \dots + n(n!) + (n+1)(n+1)!$$
$$= x_n + (n+1)(n+1)!$$
$$= [(n+1)! - 1] + (n+1)(n+1)!$$
$$= (n+2)! - 1.$$

This completes the induction step, and the result is established for all natural numbers n.

Q. 631. Show how to construct a triangle given one angle, the length of the opposite side, and the difference of the lengths of the other two sides.



Solution: The figure shows the desired triangle ABC, in which $\lfloor A$ is the given angle θ, and the lengths BC and BP are given. P is the point on AB such that *AP = *AC, so that BP is the difference in lengths of the sides from A. We show how to construct ΔABC by first constructing ΔBPC. It is easy to show that $B\hat{P}C = 90 + \frac{\theta}{2}$.

Construct BP equal to the given difference of the lengths AB and AC. Construct the ray PX by constructing the angle $90 + \frac{\theta}{2}$ for $B\hat{P}X$. The circle centre B radius the given length BC will intersect the ray PX at one point only, since $B\hat{P}X$ is an obtuse angle. This is the point C. The vertex A can finally be constructed as the point of intersection of BP produced and the perpendicular bisector of PC. (We omit the simple formal proof that the constructed triangle has all the required properties.)

Q. 632. Prove that $\sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} = 2$.

Solution: Let $\sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} = x$. Then since

$$(a+b)^3 = a^3 + 3ab(a+b) + b^3$$

we have

$$x^3 = (10 + 6\sqrt{3}) + 3\sqrt[3]{(10 + 6\sqrt{3})(10 - 6\sqrt{3})}\ x + 10 - 6\sqrt{3}$$

$$= 20 + 3\sqrt[3]{-8}\ x = 20 - 6x.$$

Now $$x^3 + 6x - 20 = 0$$

$\Rightarrow$ $$(x - 2)(x^2 + 2x + 10) = 0$$

$\Rightarrow$ $$x - 2 = 0$$

since $x^2 + 2x + 10 > 0$ for all real numbers x.

Q. 633. (a) Prove that there exist two powers of 3 having the same first 5 digits.

(b) Show that there exists a power of 3 whose first 4 digits are 1111.

Solution: (a) Since there are only 90000 different 5 digit numbers, and infinitely many powers of 3, there must be at least one 5 digit number giving the first 5 digits of infinitely many different powers of 3.

(b) Let 3^s and 3^t $(t > s)$ be powers of 3 with the same first 5 digits. Let a be the number (between 10000 & 99999) composed of those digits. Then $3^s = a \times 10^m + x$ where $0 < x < 10^m$ and $3^t = a \times 10^m + y$ where $0 < y < 10^m$.

$$\frac{a \times 10^n}{(a + 1) \times 10^m} < \frac{3^t}{3^s} < \frac{(a + 1) \times 10^n}{a \times 10^m}$$

$$(1 - \frac{1}{a + 1})\ 10^{n-m} < 3^{t-s} < (1 + \frac{1}{a}) \times 10^{n-m};$$

and since $a > 10000$

$$0.9999 \times 10^{n-m} < 3^k < 1.0001 \times 10^{n-m} \quad (\text{where } k = t - s).$$

Now divide through by 9

$$0.1111 \quad 10^{n-m} \quad 3^{k-2} \quad 11112 \quad 10^{n-m}.$$

Hence the first four digits of 3^{k-2} are 1111.

Q. 634. No three of the $n(n - 3)/2$ diagonals of a convex n-gon are concurrent at a point inside the figure. Find

(a) the number of points of intersection of diagonals inside the n-gon;

(b) the number of compartments into which the interior of the n-gon is dissected by the diagonals.

(for example, when $n = 4$ the answers are 1 and 4;
when $n = 5$ the answers are 5 and 11.)

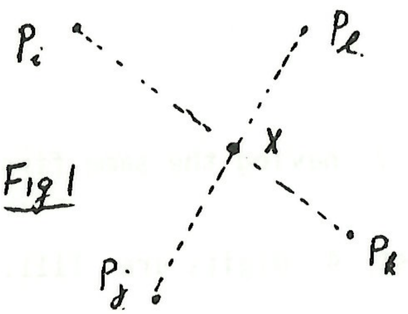


Fig 1

Solution: (a) For each selection of 4 vertices of the n-gon, there is one way to draw two diagonals which intersect inside the figure whose end-points are the chosen vertices. (See figure 1.) In fact, there is a 1 - 1 correspondence between points of intersection of diagonals inside the figure, and selections of 4 of the vertices of the n-gon. Therefore the number of intersections is equal to

$$^{n}C_4 = \frac{n(n-1)(n-2)(n-3)}{4!}.$$

(b) Let R_n be the number of regions in a dissected n-gon. Let ABC be 3 adjacent vertices. (See figure 2.) The figure can be constructed by

i) drawing the $(n - 1)$gon $ACP_1 \ldots P_{n-3}$ and all its diagonals. This produces so far R_{n-1} regions.

ii) Add B and AB, AC. The triangle ABC has been added so that the number of regions is now $R_{n-1} + 1$.

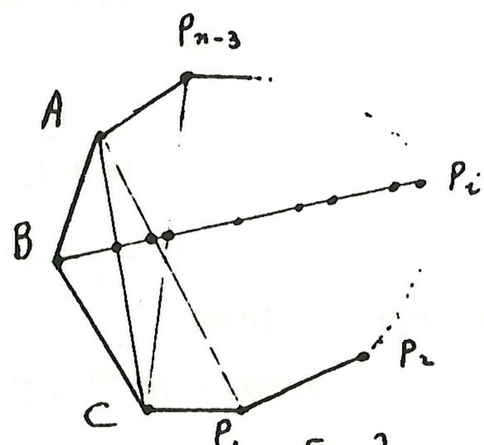


Fig 2.

iii) Draw the diagonals from B to each of the vertices $P_1, P_2, \ldots, P_{n-3}$. As each diagonal is drawn (starting from B) it cuts across previous regions. Consider the diagonal BP_i for example. If it contains r_i points of intersection with other diagonals inside the figure, it is divided into $(r_i + 1)$ segments each of which cuts a piece off a region previously present.

The number of new regions created when all the diagonals from B are drawn is thus equal to $(r_1 + 1) + (r_2 + 1) + \ldots + (r_{n-3} + 1)$. Now $r_1 + r_2 + \ldots + r_{n-3}$ is the number of extra intersections of diagonals inside the n-gon which were not obtained when the (n - 1)-gon $ACP_1 \ldots P_{n-3}$ was constructed; viz (from part (a)) ${}^nC_4 - {}^{n-1}C_4 = {}^{n-1}C_3$. We deduce that the number of additional regions created by drawing all diagonals from B is ${}^{n-1}C_3 + (n - 3)$ and that

$$R_n = R_{n-1} + 1 + {}^{n-1}C_3 + (n - 3) \quad (n > 3) \qquad (1)$$

Iterating (1) gives

$$R_n = {}^{n-1}C_3 + (n - 2) + R_{n-1} = {}^{n-1}C_3 + {}^{n-2}C_3 + (n - 2) + (n - 3) + R_{n-2} =$$

$$= [{}^{n-1}C_3 + {}^{n-2}C_3 + \ldots + {}^3C_3] + [(n - 2) + (n - 3) + \ldots + 2] + R_3.$$

Since $R_3 = 1$ we obtain $R_n = E_n + \frac{(n - 2)(n - 1)}{2}$ where $E_n = {}^{n-1}C_3 + {}^{n-2}C_3 + \ldots + {}^4C_4 + {}^3C_3$. Write ${}^3C_3 = {}^4C_4$ and use ${}^kC_3 + {}^kC_4 = {}^{k+1}C_4$ repeatedly, starting at the R.H. end of the expression for E_n. Eventually one obtains $E_n = {}^nC_4$. Therefore finally

$$R_n = {}^nC_4 + \frac{(n - 2)(n - 1)}{2}$$

$$= \frac{n(n - 1)(n - 2)(n - 3)}{4!} + \frac{(n - 2)(n - 1)}{2}$$

Q. 635. (a) Show that $5^n - 1$ is divisible by 4, n being any positive integer.

(b) A list of prime numbers $p_1, p_2, p_3, \ldots, p_n, \ldots$ is generated as follows:

$p_1 = 2$, and if $n > 1$, p_n is the largest prime factor of $p_1p_2 \dots p_{n-1} + 1$. Thus $p_2 = 3$, (the largest prime factor of $2 + 1$), $p_3 = 7$, $p_4 = 43$, $p_5 = 139$, (since $2.3.7.43 + 1 = 1807 = 13.139$.) It does not follow from this rule that p_{n+1} is larger than p_n. However, prove that the prime number 5 never occurs in the list.

Solution: (a) This is obviously true for $n = 1$. Since $5^{k+1} - 1 = 5(5^k - 1) + (5 - 1) = 5(5^k - 1) + 4$, if $5^k - 1$ is a multiple of 4, so is $5^{k+1} - 1$. By the principle of mathematical induction $5^n - 1$ is a multiple of 4 for all positive integers n.

(b) Suppose $p_n = 5$. Then $X = p_1p_2 \dots p_{n-1} + 1$ has no prime factor greater than 5. Since neither $p_1(= 2)$ nor $p_2(= 3)$ is a factor of X, X has no prime factor less than 5 either. i.e. $X = 5^m$ for some m. Note that after p_1, every p_k is odd since $p_1, p_2, \dots, p_{k-1} + 1$ is an odd number.

Therefore $X = p_1 \times \text{odd number} + 1 = 2 \times \text{odd number} + 1$

$= 2 \times (2K + 1) + 1$ for some integer K.

We obtain $5^m - 1 = 4K + 2$, which is not divisible by 4, contradicting the result proved in (a). It follows that 5 can never occur in the list.

PROBLEM SOLVERS

S. Kaye (Newington College) sent a correct solution to Q.626.

L.A. Koe (James Ruse Agricultural High School) sent good solutions of all questions (except that her solution of Q.632 was incomplete in one particular).

John Graham (St. Ignatius, Riverview) sent elegent solutions of all problems. For example, for Q.628, he observes that if ΔABC has vertices with integer co-ordinates, $\tan \hat{A} = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$ where the gradients m_1, m_2 of AB and AC are rational. Therefore it is impossible that $\tan \hat{A} = \sqrt{3}$.