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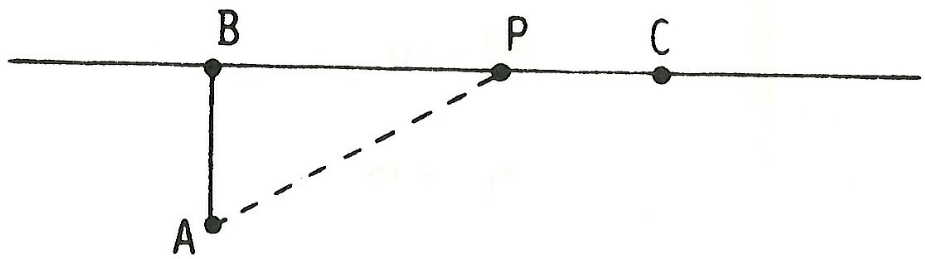
In this issue we consider some problems in the applications of mathematics found in the 1985 3 unit and 4 unit papers. Firstly two interesting turning point problems:

<u>Problem 86/1</u>. This was Q7 on the 3/4 unit paper, but is not so hard:



The diagram shows a straight road BC running due East. A four-wheel drive ambulance is in open country at A, 3 km due South of B. It must reach C, 9 km due East of B, as quickly as possible.

The driver knows that she can travel at 80 km per hour in open country and at 100 km per hour along the road. She intends to proceed in a straight line to some point P on the road and then to continue along the road to C. She wishes to choose P so that total time for the journey APC is a minimum.

a) If the distance BP is x km, derive an expression for $t(x)$, the total journey time from A to C vis P, in terms of x.

b) Show that the minimum time for the total journey APC is $6\frac{3}{4}$ minutes.

<u>Solution</u>: Note that AB = 3 km, BP = x km, and, therefore, AP = $\sqrt{(x^2 + 9)}$ km, PC = $(9 - x)$ km. The stretch AP is driven at 80 km/h, and therefore the time taken from A to P is given by $\sqrt{(x^2 + 9)}/80$ hours. Similarly, the time taken from P to C is $(9 - x)/100$ hours. Thus

$$\boxed{t(x) = \frac{\sqrt{(x^2 + 9)}}{80} + \frac{9 - x}{100}} \text{ hours.}$$

Thus $\frac{dt}{dx} = \frac{x}{80\sqrt{(x^2 + 9)}} - \frac{1}{100} = 0$ for a turning point. Let $x = x_1$ at the turning point, then, cross multiplying the above expression

$$5x_1 = 4\sqrt{(x_1^2 + 9)}.$$

Squaring both sides,

$$25x_1^2 = 16x_1^2 + 144$$

$$9x_1^2 = 144$$

$$x_1 = 4 \text{ km}.$$

When $x = x_1$, $t(x) = \frac{5}{80} + \frac{5}{100} = \frac{9}{80}$ hours $= \frac{9}{80} \times 60 = \boxed{6\frac{3}{4}}$ minutes.

We still need to show that this is the minimum time. Simply find

$$\frac{d^2t}{dx^2} = \frac{1}{80}\left[\sqrt{(x^2 + 9)} - \frac{x^2}{\sqrt{(x^2 + 9)}}\right] / (x^2 + 9)$$

$$= \frac{9}{80(x^2 + 9)^{3/2}} > 0.$$

Thus $6\frac{3}{4}$ minutes is the minimum time.

Problem 86/2. From the 4 unit paper:

A thin wire of length L is cut into two pieces, out of which a circle and a closed square are to be formed so that the sum of the areas of the circle and square so formed is a minimum. Show that this minimum value is

$$\frac{L^2}{4(\pi + 4)}.$$

Solution: Let x be the radius of the circle, and y be the side of the square. Then the sum of the circumference of the circle and the square's perimeter is

$$L = 2\pi x + 4y, \qquad (1)$$

and the sum of the area is

$$A = \pi x^2 + y^2. \qquad (2)$$

From (1), $y = \frac{1}{4}(L - 2\pi x)$. Substitute in (2),

$$A = \pi x^2 + \frac{1}{16}(L - 2\pi x)^2 = (\pi + \frac{1}{4}\pi^2)x^2 - \frac{1}{4}\pi Lx + \frac{1}{16}L^2. \qquad (3)$$

This is a parabola, whose minimum is at the vertex where $x = \frac{1}{8}\pi L/(\pi + \frac{1}{4}\pi^2) = L/2(\pi + 4)$. Substitute into (3), the minimum value of A is $L^2/4(\pi + 4)$.

We conclude this issue with the resistive motion problem, which is not quite as hard as it looks:

Problem 86/3. From the 4 unit paper:

A particle of mass 10 kg is found to experience a resistive force, in newtons, of one-tenth of the square of its velocity in metres per second, when it moves through the air.

The particle is projected vertically upwards from a point O with a velocity of u metres per second, and the point A, vertically above O, is the highest point reached by the particle before it starts to fall to the ground again. Assuming the value of g is 10 m s^{-2},

a) find the time the particle takes to reach A from O;

b) show that the height OA is $50 \log_e[1 + 10^{-3}u^2]$ metres;

c) show that the particle's velocity $w \text{ m s}^{-1}$ when it reaches O again is given by

$$w^2 = u^2(1 + 10^{-3}u^2)^{-1}.$$

Solution: a) When travelling vertically upwards, the equation of motion is

$$ma = - mg - \frac{1}{10}v^2, \qquad (1)$$

where $m = 10$ kg, $g = 10\ ms^{-2}$, a is the acceleration and v the velocity. We must remember the two expressions

$$a = \frac{dv}{dt}, \quad \text{and} \quad a = v\frac{dv}{dx}.$$

To find the time we use the first expression

$$\frac{dv}{dt} = -(10 + 10^{-2}v^2), \quad \text{or} \quad 10dt = -\frac{dv}{1 + 10^{-3}v^2}.$$

Hence since $v = u$ at $t = 0$, and $v = 0$ at $t = T$, where T is the time taken to reach 0, integration of both sides yields the result

a) $$\boxed{T = 10^{1/2}\tan^{-1}[10^{-3/2}u].\ \text{secs.}}$$

Similarly, for (b), we are required to solve the equation

$$v\frac{dv}{dx} = -10(1 + 10^{-3}v^2)$$

i.e.

$$-10dx = \frac{vdv}{1 + 10^{-3}v^2}.$$

Let X be the distance $0A$, then, intergrating both sides

$$-10\int_0^X dx = \int_u^0 \frac{vdv}{1 + 10^{-3}v^2} = -\frac{1}{2}10^3\log_e[1 + 10^{-3}u^2]$$

b) therefore

$$\boxed{X = 50\log_e[1 + 10^{-3}u^2].} \qquad (2)$$

On the way down, the equation to be satisfied is

$$10\, v\frac{dv}{dx} = 10g - \frac{1}{10}v^2 \qquad (3)$$

or

$$v\frac{dv}{dx} = 10(1 - 10^{-3}v^2).$$

$$-10 \int_x^0 dx = \int_0^w \frac{vdv}{1 - 10^{-3}v^2} \quad \text{which yields}$$

$$X = -50 \log_e[1 - 10^{-3}w^2]. \qquad (4)$$

Comparing (2) and (4),

$$1 - 10^{-3}w^2 = \frac{1}{1 + 10^{-3}u^2},$$

or

c) $$\boxed{w^2 = u^2(1 + 10^{-3}u^2)^{-1}}.$$

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