SOLUTIONS TO PROBLEMS FROM VOLUME 21, NUMBER 3

(Problems in volume 21, number 3 appeared incorrectly numbered as 536 - 546, instead of 636 - 646. We correct the numbering in the solutions.)

Q. 636 In the play off match for the chess club championship between the three players who had finished level after the preliminary tournament, each pair played the same number of games. Reporting on the match, the club news sheet stated "A won most games, B lost fewest times, but C won the championship with the most points". Could the report be correct?

Solution: Assuming the usual scoring of 1 point for a win, $\frac{1}{2}$ for a draw, 0 for a loss the report could well be correct. For example, if each pair played 7 games resulting

A v B : 2 wins, 2 losses, 3 draws
A v C : 3 wins, 4 losses
B v C : 7 draws.

Then A wins 2 + 3 times, B wins 2 + 0 times, C wins 4 + 0 times.
A loses 2 + 4 times, B loses 2 + 0 times, C loses 3 + 0 times.

Points scored A : $3\frac{1}{2} + 3 = 6\frac{1}{2}$

B : $3\frac{1}{2} + 3\frac{1}{2} = 7$

C : $4 + 3\frac{1}{2} = 7\frac{1}{2}$.

(Obviously there is no limit to the number of possible alternatives, since the above results could all be multiplied by the same factor, and/or any additional number of "rounds" resulting in draws could be played.)

Correct solution from L-A. Koe (James Ruse Agricultural College).

Q. 637 Is it possible to dissect a triangle into concave pentagons? (i.e. pentagons with re-entrant angles).

Solution: The figure shows how any triangle can be dissected into 2 or into 3 re-entrant pentagons.

Figure 1.

Since a triangle can be subdivided into n triangles by joining a vertex to (n - 1) points in the opposite side, it is easy to see that a triangle can be dissected into m re-entrant pentagons for any whole number m greater than 1. (A much more challenging problem is to dissect a triangle into convex pentagons. The minimum number M of pentagons is not more than 9, as the accompanying diagram (figure 2) shows. I do not know if there is a dissection into fewer than 9 pentagons. Anyone care to establish the correct value of M?)

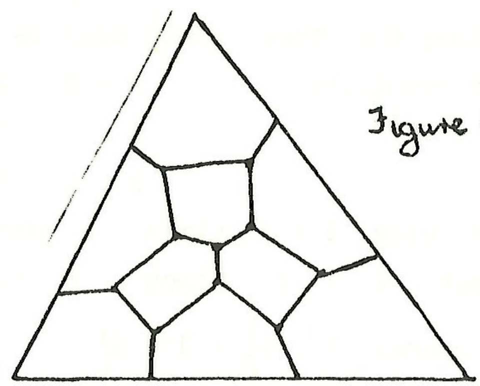


Figure 2.

Correct solution from S. Kaye (Newington College).

Q. 638 Tiddlyball is a three player game. In each round, the winner scores a point, the runner up is awarded b points, and the loser gets c points, where $a > b > c$ are positive integers. One day Xavier, Yvonne and Zachary played some tiddlyball and the final score was

Xavier - 20 Yvonne - 10 Zachary - 9.

Yvonne won the second round. Who won the first round, and how many points did Zachary get in the last round?

Solution: Since the total number of points scored is 39, there must have been 3 rounds played with $a + b + c = 13$. (Note $a + b + c > 3$) Yvonne's score, including an a, totalled fewer than 13 points. She must have come third in the other two rounds, and $a + 2c = 10$. We deduce that $b = c + 3$.

Since Zachary's average score per round exceeds 6, we must have $a > 7$. $2c < 3$ yielding $c = 1$ as the only possibility (since c is a positive integer). Therefore $a = 8$, $b = 4$, $c = 1$. Xavier must have won the 1st and 3rd rounds and come second in the 2nd round. Hence Zachary must have come third in the 2nd round, and second in the first round and also the last round, in which he scored 4 points.

Correct solutions from L-A. Koe (James Ruse) and S. Kaye (Newington).

Q. 639 For a positive integer n, let n^* be the number which results by writing n to the base 2 and then reading the result as though it were written in base 3. For instance, if $n = 6$, then $6 = 110_2$ and $110_3 = 12$ so that $6^* = 12$. Find all numbers n such that $n^* = 9n$ and prove there are no others.

Solution: Let $n = a_0 + a_1 \times 2 + a_2 \times 2^2 + \ldots + a_n \times 2^n$ where each a_k is either 0 or 1. Then $n^* = a_0 + a_1 \times 3 + a_2 \times 3^2 + \ldots + a_n \times 3^n$. Now n^* is divisible by 9 only if $a_0 = a_1 = 0$. Hence if $n^* = 9n$ we have

$$9 \times (a_2 2^2 + a_3 \times 2^3 + \ldots + a_n \times 2^n) = 3^2(a_2 + a_3 \times 3 + \ldots + a_n \times 3^{n-1})$$

which after transposing terms gives

$$3a_2 + 5a_3 + 7a_4 + 5a_5 = 17a_6 + (3^5 - 2^7)a_7 + \ldots .$$

The coeffecent of a_k on the R.H.S. is $3^{k-2} - 2^k$. Observe (it is easy to give a formal proof) that these coefficients increase rapidly ($3^5 - 2^7$ is already equal to 115). Since the L.H.S. is at most $3 + 5 + 7 + 5 = 20$ we must have $a_k = 0$ for $k > 6$, $a_6 = 1$, and

$$3a_2 + 5a_3 + 7a_4 + 5a_5 = 17.$$

Clearly the only solution is $a_2 = 0$, $a_3 = a_4 = a_5 = 1$.

This yields $n = 2^3 + 2^4 + 2^5 + 2^6 = 120$ and

$$n^* = 3^3 + 3^4 + 3^5 + 3^6 = 1080, \text{ the only solution.}$$

Correct solution from L-A. Koe (James Ruse).

Q. 640 The numbers 1, 2, ..., 64 are written onto the squares of an 8×8 chessboard. Prove that there is a pair of adjacent squares which contain numbers that differ by $\leqslant 16$. (Squares are adjacent if they share a corner, or a side.)

Solution: Divide the board into quarters in both directions obtaining 16 "cells" each consisting of 4 mutually adjacent squares. ( : one cell.) Of the seventeen numbers 1,2,3, ..., 17 there must be more than one in at least one of the cells, (Pigeon hole principle). The difference of any two of these numbers is at most 16. Q.E.D.

Comment Of course, this does not prove that it is possible to place the numbers on the board in such a way that the minimum difference is equal to 16. In fact, it is not too difficult to show that it must actually be less than 16. The "best" value for the minimum difference is thus 15, since this can be achieved by trial.

Correct solution from L-A. Koe (James Ruse).

Q. 641 Does there exist an infinite sequence of perfect squares such that the sum of the first n elements is also a perfect square for every n?

Solution From the identity

$$(2u + 1)^2 + [2u(u + 1)]^2 = [2u(u + 1) + 1]^2$$

it is evident that given any odd perfect square $(2u + 1)^2$ greater than 1, we can always find an even square which added to it yields a larger odd square. We can use this to construct a sequence such as

$$3^2\ 4^2,\ 12^2,\ \ldots\ .$$

having the required property. Since $3^2 + 4^2 + 12^2 = 13^2 = (2 \times 6 + 1)^2$, our construction would yield $[2 \times 6 \times (6 + 1)]^2 = 84^2$ for the next term, the sum of the first four terms being then 85^2. [The term after 84^2 would similarly be $(2 \times 42 \times 43)^2$].

Correct solution from L-A. Koe (James Ruse).

Q. 642 A certain pentagon has the property that each of the 5 lines joining a vertex to the mid-point of the opposite side bisects the pentagon into two parts of equal area. Prove that these five lines are concurrent.

Solution: In the figure M, N, and P are the mid points of CD, BC and DE respectively. It is obviously sufficient to prove that AM, BP and EN are concurrent. Observe that

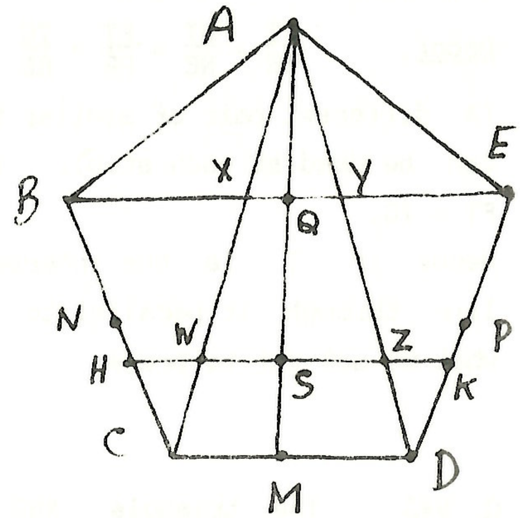


Figure 1.

1) All the triangles cut off the pentagon by a diagonal have the same area. e.g. $\Delta ABC = \Delta ADE$.

Proof. For area ΔABC = area ABCM − area ΔACM
= area AEDM − area ΔADM
= area ΔADE.

2) Every diagonal is parellel to one of the sides of the pentagon e.g. $BE \parallel CD$.

Proof. From (i), area ΔBCD = area ΔCDE. The result $BE \parallel CD$ follows immediately since the triangles stand on the same base CD.

3) Q is the mid point of BE.

Proof. Since $XY \parallel CD$ one can easily prove from $\frac{XQ}{CM} = \frac{AQ}{AM} = \frac{YQ}{DM}$ that $XQ = YQ$. The desired result therefore follows if we can show $BX = YE$. But

$$\frac{BX}{YE} = \frac{\text{area } \Delta ABX}{\text{area } \Delta AYE} \quad \text{and} \quad \frac{BX}{YE} = \frac{\text{area } \Delta CBX}{\text{area } \Delta DYE} \quad (\text{since } BE \parallel CD)$$

therefore

$$\frac{BX}{YE} = \frac{\text{area } \Delta ABX + \text{area } \Delta CBX}{\text{area } \Delta AYE + \text{area } \Delta DYE} = 1 \quad (\text{from } (1))$$

4) Similarly AM bisects any other interval paralled to CD whose end points lie on the pentagon. e.g. $HS = SK$.

Proof. Because $BE \parallel HK \parallel CD$ we have

$$\frac{HW}{BX} = \frac{CH}{CB} = \frac{DK}{DE} = \frac{ZK}{YE}.$$

Since $BX = YE$ we deduce $HW = ZK$. Also $WS = SZ$. Therefore $HS = HW + WS$ $SZ + ZK = SK$.

5) In any trapezium NPEB the line interval FTG parallel to NP and BE is bisected at T, the intersection of the diagonals NE and BP.

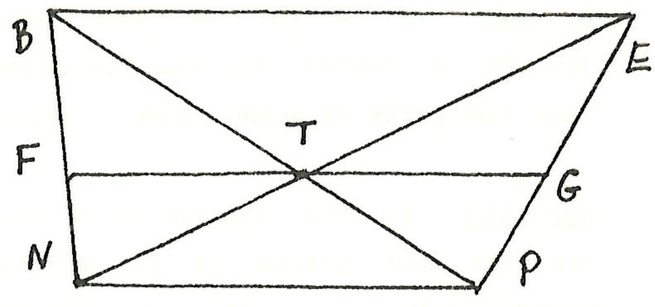


Figure 2.

__Proof.__ $\frac{FT}{BE} = \frac{NT}{NE} = \frac{PT}{PB} = \frac{TG}{BE}$

(A different pair of similar triangles can be used at each step). Therefore FT = TG.

Hence if T is the intersection of BP and EN in figure 1, it bisects the line through it parallel to CD and therefore lies on AM by (4). This is what we set out to prove.

__Q. 643__ The triangle ABC is right angled at A, and the lengths of AB, AC are 3,4 respectively. It is possible to draw two squares inside the triangle all of whose vertices lie on the sides of the triangle. Find the area of the overlap region inside both squares.

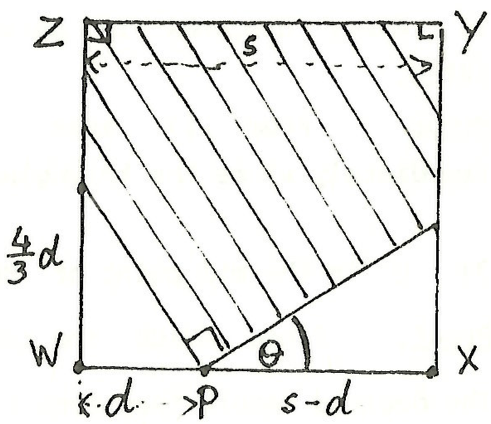


Figure 1.

__Solution:__ In figure 1, where $\tan\theta = \frac{3}{4}$, the shaded area is

$$A = s^2 - \frac{1}{2}d \times \frac{4}{3}d - \frac{1}{2}(s - d) \times \frac{3}{4}(s - d)$$

$$= \frac{5}{8}s^2 - \frac{25}{24}d^2 + \frac{3}{4}s.d. \qquad (1)$$

where s is the length of a side of the square WXYZ, and d is the length WP.

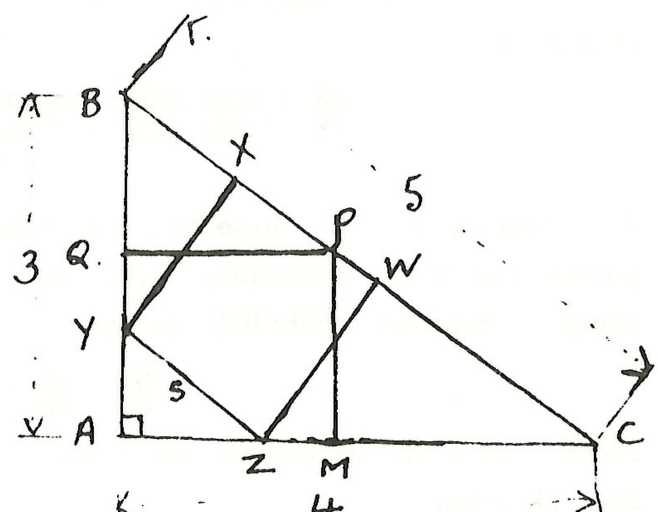


Figure 2.

In figure 2, the two squares are shown in the triangle. It is easy to see that if YZ = s, then $WC = \frac{4}{3}s$, $BX = \frac{3}{4}s$, WX = S whence $(\frac{3}{4} + 1 + \frac{4}{3})s = BC = 5$ yielding $s = \frac{60}{37}$.

If CP = x, then $PM = PQ = \frac{3}{5}x$ and

$$BP = \frac{5}{4} \times \frac{3}{5}x = \frac{3}{4}x.$$

Hence $x + \frac{3}{4}x = BC \cdot 5$ yielding $x = \frac{4}{7} \times 5$.

Then $d = WP = CP \quad CW = x \quad \frac{4}{3}s = \frac{9 \times 20}{7 \times 37}$. Substituting these values of s and d into (1) gives

$$A = \frac{60^2}{7^2 \times 37} \quad 1.986$$

for the overlap area.

Correct solution from S. Kaye (Newington) and L A Koe James Ruse)

Q. 644 Let S be a set of m real numbers $m \geq 2$) with the property that the absolute value of each number in S is not less than the absolute value of the sum of all the others. Prove that the sum of all the numbers in S is zero.

Solution: We first rule out the possibility that all nonzero numbers in S have the same sign (except, of course, when all members of S are equal to zero). For if s_0 was then the numerically smallest member of S $|s_0| \leq |s|$ for all other elements s of S, with at least one $|s| > 0$ and $|s_0| < \sum_{\substack{s \in S \\ s \neq s_0}} |s| = |\sum s|$, contradicting the data. Thus if the data is satisfied and not all elements of s are zero then S contains both positive and negative elements. Now suppose $\sum_{s \in S} s = k \neq 0$. Find $s_0 \in S$ with sign opposite to that of k. Then

$$|k - s_0| = |k| + |s_0| > |s_0|$$

i.e. $|\sum_{\substack{s \in S \\ s \neq s_0}} s| > |s|$ contradicting the data.

Thus it is impossible that $k \neq 0$. Q.E.D.

Correct solution from L-A. Koe (James Ruse).

Q. 645 Let $\square$ be an operation which combines two integers to give a new integer. Suppose that

$$x \square (y + z) = (y \square x) + (z \square x)$$

for all x, y, z. Prove that

$$u \square v = v \square u \quad \text{for all} \quad u, v.$$

Solution: Put $y = x$, $z = 0$ in the given identity, obtaining

$$x \square (x + 0) = x \square x + 0 \square x$$

therefore

$$x \square x = x \square x + 0 \square x$$

whence

$$0 \square x = 0 \quad \text{for any} \quad x. \qquad (1)$$

Now, put $z = 0$, $y = u$, $y = v$ in the given identity, obtaining

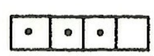
$$u \square (v + 0) = v \square u + 0 \square u = v \square u$$

using (1) i.e.

$$u \square v = v \square u.$$

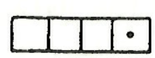
Correct solution from L-A. Koe (James Ruse).

Q. 646 I put 62 stones on an ordinary 8×8 chessboard putting one in each square and leaving two opposite corners empty. I now remove stones from the board by jumping two at a time and removing the two jumped over. Thus if I have



a b c d

I can jump the stone in square a onto d, removing b and c and leaving



Jumps can be horizontal or vertical but not diagonal. Prove that if I keep jumping until there are just two stones left, they must be on squares of the same colour.

Solution: The opposite corner squares are both of the same colour; say white. Then we start with 32 stones on black squares and 30 stones on white squares. At each jump two stones on adjacent squares are removed, but also the stone which does the jump finishes on a square of the opposite colour. The net effect is to remove either 2 "black" stones or 2 "white" stones. If before the move there are even numbers of both categories of stones, the same is true after the jump. Hence there will never be an odd number of "white" or of "black" stones at any stage. In particular no position with exactly one white stone or one black stone can ever be reached.

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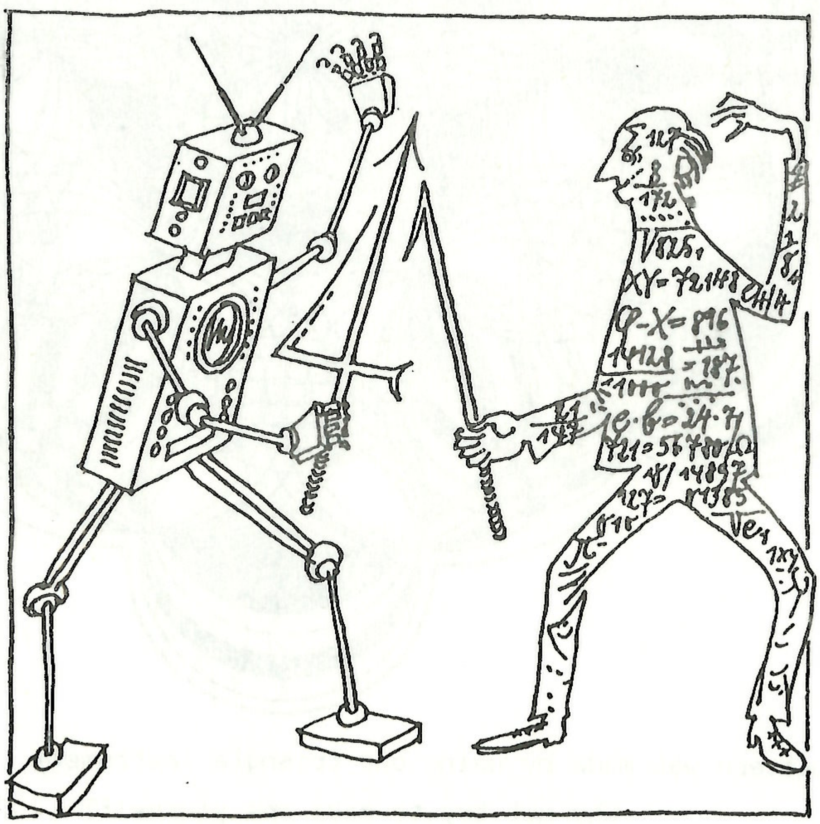
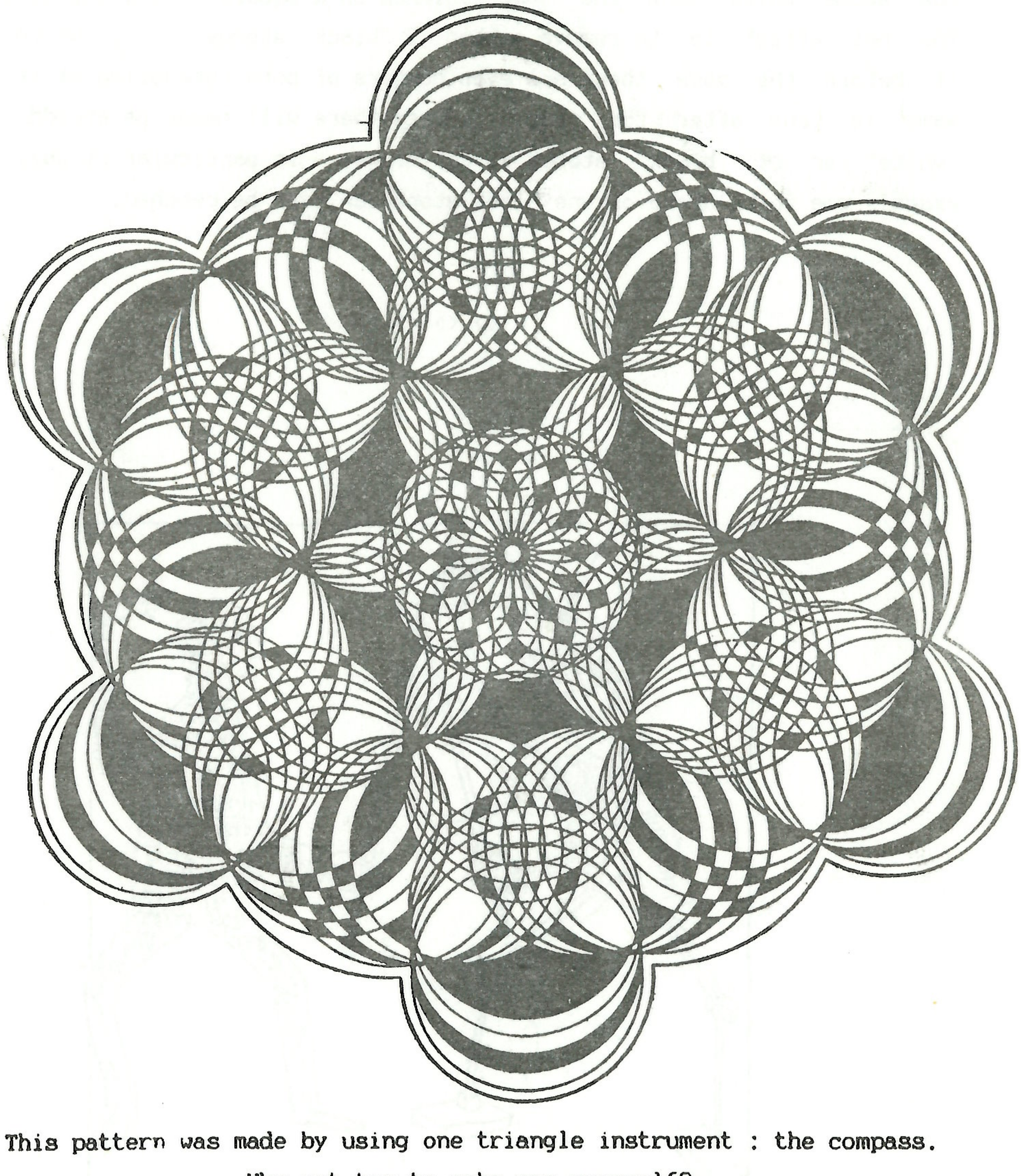


Today's rivals??

Circles and circles and circles ...



This pattern was made by using one triangle instrument : the compass.
Why not try to make one yourself?

[Reproduced from: Kozépiskolai Matematikai Lapok, 1985 10.no.]