Parabola Volume 22, Issue 2 (1986)

THE 1986 SCHOOL MATHEMATICS COMPETITION

JUNIOR DIVISION

1. Linus has a litre flask of pure orange juice and an empty litre flask. He pours some juice into the empty flask, tops up with water, stirs the mixture and uses it to top up the first flask. Show that this flask contains at least 75% orange juice.

Solution: Suppose Linus pours x litres of orange juice from the full flask A into the empty flask B. After topping up with water, flask B contains x litres of orange juice and $1 - x$ litres of water, that is a ratio of $x : 1 - x$ of orange juice to water. Thus x litres of this mixture will contain x^2 litres of orange juice and $x(1 - x)$ litres of water. If this x litres is used to top up flask A, this flask will now contain $(1 - x) + x^2$ litres of orange juice and $x(1 - x)$ litres of water. We want to choose x with $0 \leq x \leq 1$ so that $x^2 - x + 1$ is as small as possible. Since $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4}$, the minimum occurs when $x = \frac{1}{2}$ and $x^2 - x + 1 = \frac{3}{4}$. For any x with $0 \leq x \leq 1$, we have $x^2 - x + 1 \geq \frac{3}{4}$, so flask A contains at least $\frac{3}{4}$(= 75%) orange juice at the end.

2. In the magic square shown, the sums of the numbers in each row, in each column and in each of the two main diagonals are all equal to 15.

8	1	6
3	5	7
4	9	2

a) Find distinct positive integers a,b,c,d,e,f,g,h,k so that the products of the numbers in each row, in each column and in each of the two main diagonals of the square shown are equal.

a	b	c
d	e	f
g	h	k

b) Given b = 1, what is the smallest integer which can be taken for e in such a magic product square containing distinct positive integers? Prove your result.

Solution: a) The easiest answer comes from using the given magic square with all sums equal to 15. Take $a = 2^8$, $b = 2^1$, $c = 2^6$, $d = 2^3$, $e = 2^5$, $f = 2^7$, $g = 2^4$, $h = 2^9$, $k = 2^2$. Then all products are equal to 2^{15}.

b) Suppose $b = 1$. The products of the top row and the diagonal are equal, so $c = ek$; the products of the top row and the diagonal are equal so $a = eg$. Thus the common product is $abc = e^2gk$. The product of the bottom row is ghk and since this is also e^2gk, we get $h = e^2$. The product of the middle column is now e^3 and since this equals e^2gk we have $e = gk$. We can now fill in $a = g^2k$, $b = 1$, $c = gk^2$ $d = k^2$, $e = gk$, $f = g^2$ and $h = g^2k^2$ and this gives a solution with common product g^3k^3 for any positive integers g and k. To get distinct positive integers, we must take $g \geq 2$, $k \geq 2$, $g \neq k$, so the smallest choice for e is 6 coming from $g = 2$, $k = 3$.

256	2	64
8	32	128
16	512	4

12	1	18
9	6	4
2	36	3

also

50	1	20
4	10	25
5	100	2

etc.

p^2q	1	pq^2
q^2	pq	p^2
p	p^2q^2	q

3. The numbers $b_1, b_2, b_3, \ldots$ in binary notation are given by $b_1 = 111$, $b_2 = 1101$, $b_3 = 11001$, $b_4 = 110001$, and so on. (That is, $b_1 = 2^2 + 2 + 1 =$ seven, $b_2 = 2^3 + 2^2 + 1 =$ thirteen, and so on.) How many of these numbers are squares? Prove your assertion.

Solution: The number $b_n = 1100 \ldots 01$ with $n - 1$ zeros in binary notation is $b_n = 2^{n+1} + 2^n + 1 = 3.2^n + 1$. If $b_n = m^2$, say, then

$$3.2^n = m^2 - 1 = (m - 1)(m + 1) \qquad (*)$$

Now $n = 1$ does not give a solution because $b_1 = 7$ is not a square, so $n \geq 2$. The factors $m - 1$ and $m + 1$ in (*) differ by 2, so they must both be even because their product is divisible by 2^n. In fact, one of $m - 1$ and $m + 1$ is divisible by 2 but by no higher power of 2, and the other is divisible by 2^{n-1}. (If $m - 1$ and $m + 1$ were both divisible by 2^2, then $2 = (m + 1) - (m - 1)$ would also be divisible by 2^2 which is nonsense.) The only other prime dividing $(m - 1)(m + 1)$ is 3, so one of $m - 1$ and $m + 1$ is divisible by 3 and there are no other primes dividing $m - 1$ and $m + 1$. This gives four possibilities:

$m - 1 = 2$, $m + 1 = 3.2^{n-1}$; $\qquad$ $m + 1 = 2$, $m - 1 = 3.2^{n-1}$;

$m - 1 = 2.3$, $m + 1 = 2^{n-1}$; $\qquad$ $m + 1 = 2.3$, $m - 1 = 2^{n-1}$.

The first two possibilities do not lead to any solutions. The third gives $m = 7$, $n = 4$, $b_4 = 2^5 + 2^4 + 1 = 7^2$, and the fourth gives $m = 5$, $n = 3$, $b_3 = 2^4 + 2^3 + 1 = 5^2$. So just two of the b_n are squares.

4. Tweedledum and Tweedledee play the following game. They place 10 tickets numbered 1 to 10 in a jar and they draw two tickets from the jar. If the sum of the two numbers drawn is odd then Tweedledum wins; if the sum is even then Tweedledee wins. Is it a fair game?

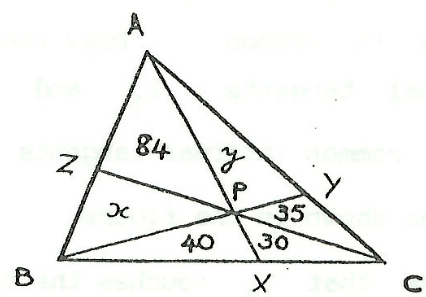
What if they play the same game with 1986 tickets numbered 1 to 1986?

<u>Solution</u>: With 10 tickets there are $\frac{10 \times 9}{2}$ possible pairs of tickets that can be drawn from the jar. There are 5 even numbered tickets, so there are $\frac{5 \times 4}{2}$ possible pairs of tickets which are both even. Similarly, there are $\frac{5 \times 4}{2}$ possible pairs of tickets which are both odd. This gives 5×4 possible pairs of tickets with an even sum. There are 5×5 possible pairs of tickets one of which is odd and one of which is even, that is 5×5 possible pairs of tickets with an odd sum. So the probability that Tweedledum wins the game is $\frac{5 \times 5}{\frac{1}{2} \times 10 \times 9} = \frac{5}{9}$ and the probability that Tweedledee wins is $\frac{5 \times 4}{\frac{1}{2} \times 10 \times 9} = \frac{4}{9}$. The game is biased in favour of Tweedledum.

With 1986 tickets, there are $\frac{1986 \times 1985}{2}$ possible pairs of tickets, 993×992 pairs with an even sum and 993×993 with an odd sum. The probability that Tweedledum wins is $\frac{993}{1985}$ and the probability that Tweedledee wins is $\frac{992}{1985}$. The game is still biased in favour of Tweedledum.

(See also Senior Division, question 1.).

5. Let P be a point inside the triangle ABC and divide the triangle into six pieces as shown. The areas of four of the pieces are 40, 30, 35 and 84 as shown. Find the area of the triangle ABC.



Solution: Label the areas of the unknown triangles x and y as shown. We use the fact that the ratio of the areas of two triangles with the same height is the same as the ratio of their bases. Thus

$$\frac{\text{area } \Delta BPX}{\text{area } \Delta CPX} = \frac{BX}{XC}, \quad \frac{\text{area } \Delta BAX}{\text{area } \Delta CAX} = \frac{BX}{XC}.$$

But area $\Delta BPX = 40$, area $\Delta CPX = 30$, area $\Delta BAX = 40 + 84 + x$, area $\Delta CAX = 30 + 35 + y$, so we have the equation

$$\frac{40}{30} = \frac{124 + x}{65 + y}, \quad \text{i.e.} \quad 30x - 40y = 2600 - 3720 = -1120. \qquad (1)$$

Apply the same reasoning to the triangles APY, CPY and ABY, CBY on the base AC. This gives

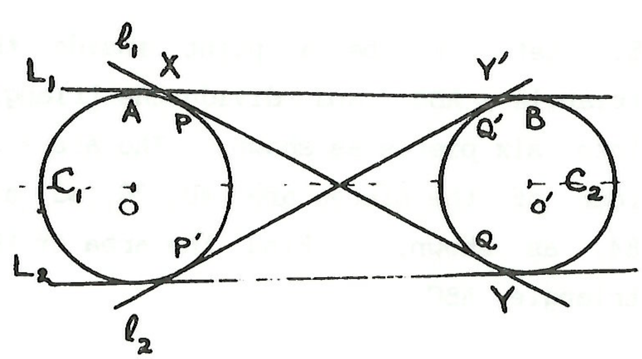
$$\frac{y}{35} = \frac{x + y + 84}{105}, \quad \text{i.e.} \quad 35x - 70y = -35 \times 84. \qquad (2)$$

dividing (1) by 10 and (2) by 35 gives the equations

$$3x - 4y = -112, \qquad x - 2y = -84.$$

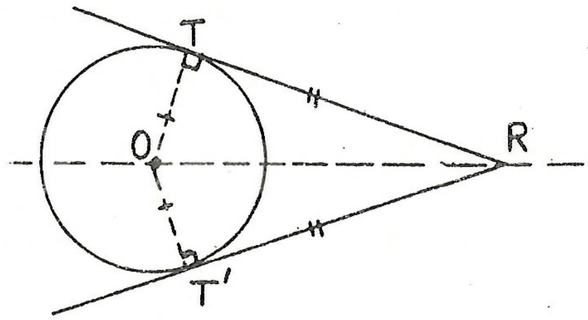
Solving simultaneously gives the solution $x = 56$, $y = 70$. So the area of ΔABC is 315.

6. Let C_1 and C_2 be circles with no points in common. Draw their common external tangents, L_1 and L_2, and their common internal tangents, ℓ_1 and ℓ_2, as shown in the figure. Suppose that L_1 touches the circles at A and B and that ℓ_1 intersects L_1 and L_2 at X and Y as shown. Prove that AB and XY are equal in length.



Solution: Label the figure as shown. (The common tangent ℓ_1 touches the two circles at P and Q and the common tangent ℓ_2 touches the circles at P′ and Q′ and ℓ_2 and L_1 meet at Y′.)

We will use many times the fact that the two tangents from an external point to a circle are equal in length. (This follows by symmetry; since the figure is symmetrical in the line OR, or from the fact that triangles OTR, OT′R are congruent.) We also use the fact that the figure in the problem is symmetrical in the line OO′ joining the centres of the two circles.



Now, XA = XP = a (say) because these are tangents from X to C_1; and Y′B = Y′Q′ = b (say) because these are tangents from Y′ to C_2. Also, PQ = P′Q′ = c (say), from the symmetry in the line joining the centres, and YQ = Y′Q′ = b for the same reason. Put XY′ = d.

Compare the tangents XQ and XB from X to C_2:

$$XQ = XP + PQ = a + c, \quad XB = XY' + Y'B = d + b, \quad \text{so} \quad a + c = d + b. \quad (1)$$

Compare the tangents Y′P′ and Y′A from Y′ to C_1:

$$Y'P' = Y'Q' + Q'P' = b + c, \quad Y'A = Y'X + XA = d + a, \quad \text{so} \quad b + c = d + a. \quad (2)$$

From (1), $a - b = d - c$; from (2), $a - b = c - d$. Consequently, $a = b$ and $c = d$ and finally,

$$AB = AX + XY' + Y'B = a + d + b = a + c + b = XP + PQ + QY = XY.$$

7. One morning a train left Central at the published time and travelled the 8 kilometres to Ashfield at an average speed of 33 kilometres per hour, arriving just as the minute hand covered the hour hand on the old station clock. What time did it show on the engine driver's digital watch.

Solution: First, we find the times when the minute hand on the old station clock covers the hour hand. Suppose the time is m minutes after 12 o'clock. The hour hand has moved through $\frac{m}{60} \times 5 = \frac{m}{12}$ divisions on the clock face (divided into 60 divisions). The minute hand has made r complete revolutions and then moved through $m - 60r$ divisions on its next revolution. Here r is one of the integers 0,1,2, ..., 11. We want $\frac{m}{12} = m - 60r$, that is $11m = 720r$. This gives 11 possible times:

$r = 0$	$m = 0$	time 12 o'clock
$r = 1$	$m = \frac{720}{11} = 65\frac{5}{11}$	$1{:}05\frac{5}{11}$
$r = 2$	$m = \frac{1440}{11} = 130\frac{10}{11}$	$2{:}10\frac{10}{11}$
⋮		
$r = 10$	$m = \frac{7200}{11} = 654\frac{6}{11}$	$10{:}54\frac{6}{11}$

The successive times differ by $65\frac{5}{11}$ minutes.

The train journey starts at the published time which is an exact number of minutes after 12 o'clock. The journey takes $\frac{8}{33} \times 60 = \frac{160}{11} = 14\frac{6}{11}$ minutes. The finishing time must be one of the times in the list above which contains the fraction $\frac{6}{11}$ minutes. The fractions in this list are all different, so the time we want $10{:}54\frac{6}{11}$. On a digital watch, this would appear as 10:54 a.m. or 10:54:~~55~~33 a.m. (We are told all this happened in the morning.)

SENIOR DIVISION

1. Tweedledum and Tweedledee play the following game. They place 10 tickets numbered 1 to 10 in a jar and they draw two tickets from the jar. If the sum of the two numbers drawn is odd then Tweedledum wins; if the sum is even then Tweedledee wins. Is it a fair game?
What if they play the same game with 1986 tickets numbered 1 to 1986?
What if they draw three tickets from the jar of 1986 tickets and decide the winner by the same rule?

Solution: For the case when two tickets are drawn see Junior Division question 4. Now suppose they draw three tickets. Tweedledum wins if there is one odd number or three odd numbers in the tickets drawn. Tweedledee wins if there is one even number or three even numbers in the tickets drawn. Since the jar initially contains the same number of odd and even tickets, this must be a fair game.
What happens with 1987 tickets?

2. Suppose that the integers x, y and z have greatest common divisor 1 and satisfy $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$. (For example $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.) Show that $x + y$ is a square.

Solution: Let d be the greater common divisor of x and y, so that $x = dx'$ and $y = dy'$ and the integers x' and y' are relatively prime. Since x, y and z have no common factors, d and z must be relatively prime. Now $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$, so $z(x + y) = xy$, that is $z(x' + y') = dx'y'$. In this equation, d and z are relatively prime, so d divides $x' + y'$. Also, x' and y' are relatively prime, so x' and $x' + y'$ are relatively prime, so x' divides z. Similarly y' divides z. So we must have $z = x'y'$ and $x' + y' = d$. Thus $x + y = d^2$ is always a square.

3. A kangaroo hops around the vertices of a regular tetrahedron. When he is at a vertex, he chooses one of the three other vertices with equal probability and jumps there. Given that he starts at vertex A and jumps seven times, find the probability that he finishes at A.

What is the probability that he finishes at A after a very large number of jumps?

Solution: Let p_n be the probability that the kangaroo is at A after n jumps. Since he starts at A, $p_0 = 1$. The first jump takes him away from A, so $p_1 = 0$. At the second jump, he returns to A with probability $\frac{1}{3}$, so $p_2 = \frac{1}{3}$. For the third jump, there are two possibilities. If he is at A, then he jumps away from A; if he is not at A, he has probability $\frac{1}{3}$ of jumping to A. The second alternative occurs with probability $1 - p_2 = \frac{2}{3}$. So $p_3 = (1 - p_2)\frac{1}{3} = \frac{2}{9}$. In the same way, $p_{n+1} = (1 - p_n)\frac{1}{3}$ for any $n \geq 0$. Now we can calculate

$$p_4 = \frac{7}{27}, \quad p_5 = \frac{20}{81}, \quad p_6 = \frac{61}{243}, \quad p_7 = \frac{182}{729}.$$

So the probability that he is at A after 7 jumps is $\frac{182}{729}$.

The calculations suggest that $p_n \to \frac{1}{4}$ as $n \to \infty$. This is reasonable because there are 4 available vertices and the kangaroo should spend about the same time at each one. In fact, $p_{n+1} - \frac{1}{4} = (1 - p_n)\frac{1}{3} - \frac{1}{4} = -\frac{1}{3}(p_n - \frac{1}{4})$, which shows that $p_n - \frac{1}{4} = (-\frac{1}{3})^n(p_0 - \frac{1}{4}) = \frac{3}{4}(-\frac{1}{3})^n$.

(Use induction: $p_1 - \frac{1}{4} = -\frac{1}{3}(p_0 - \frac{1}{4})$, $p_2 - \frac{1}{4} = -\frac{1}{3}(p_1 - \frac{1}{4}) = (-\frac{1}{3})^2(p_0 - \frac{1}{4})$, etc.)

So $p_n = \frac{1}{4} + \frac{3}{4}(-\frac{1}{3})^n$ and the probability that the kangaroo is at A after n steps is very close to $\frac{1}{4}$ if n is large.

4. The function f is defined on the positive integers and satisfies

$$f(1) = 1,$$
$$f(2n) = 2f(n) + 1 \quad (n \geq 1),$$
$$f(f(n)) = 4n + 1 \quad (n \geq 2).$$

Find f(1986).

<u>Solution</u>: We can calculate f(1986) as follows. First f(1) = 1. Next, from f(2n) = 2f(n) + 1,

$$f(2) = 3, \quad f(4) = 7, \quad f(8) = 15, \quad f(16) = 31.$$

From f(f(n)) = 4n + 1,

$$f(31) = f(f(16)) = 4 \times 16 + 1 = 65.$$

From f(2n) = 2f(n) + 1,

$$f(62) = 131, \quad f(124) = 263, \quad f(248) = 527.$$

From f(f(n)) = 4n + 1 again,

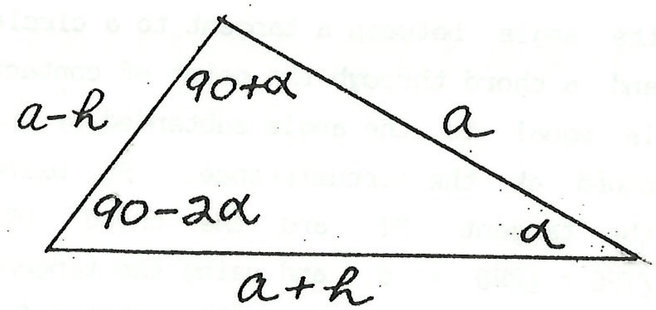
$$f(527) = f(f(248)) = 4 \times 248 + 1 = 993,$$
$$f(993) = f(f(527)) = 4 \times 527 + 1 = 2109.$$

Finally from f(2n) = f(n) + 1,

$$f(1986) = 4219.$$

(Of course, you should discover this happy chain of events by working backwards from $f(1986)$. To find $f(1986)$, we need to have $f(993)$ and then we can use the second equation in the problem. To find $f(993)$, we can use the third equation, if we can find such an n_1 with $f(n_1) = 993$. But $993 = 4 \times 248 + 1$, so we can find such an n_1 by putting $n = 248$ in the third equation. This gives $f(f(248)) = 993$, so $n_1 = f(248)$. Now we try to calculate $f(248)$ by the same method.)

5. The lengths of the sides of a triangle are in arithmetic progression and the greatest angle exceeds the least angle by 90^o. Find the ratio of the lengths of the sides.



Solution: Consider the triangle with sides and angles as shown. By the sine rule,

$$\frac{a}{\sin(90-2\alpha)} = \frac{a-h}{\sin\alpha} = \frac{a+h}{\sin(90+\alpha)},$$

that is

$$\frac{a}{\cos 2\alpha} = \frac{a-h}{\sin\alpha} = \frac{a+h}{\cos\alpha}. \qquad (*)$$

The second and third terms give $\tan\alpha = \frac{a-h}{a+h}$. Now

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{(a+h)^2-(a-h)^2}{(a+h)^2+(a-h)^2} = \frac{2ah}{a^2+h^2}.$$

So the equations (*) give

$$\sin\alpha = \frac{(a-h)\cos 2\alpha}{a} = \frac{2h(a-h)}{a^2+h^2}, \quad \cos\alpha = \frac{(a+h)\cos 2\alpha}{a} = \frac{2h(a+h)}{a^2+h^2}.$$

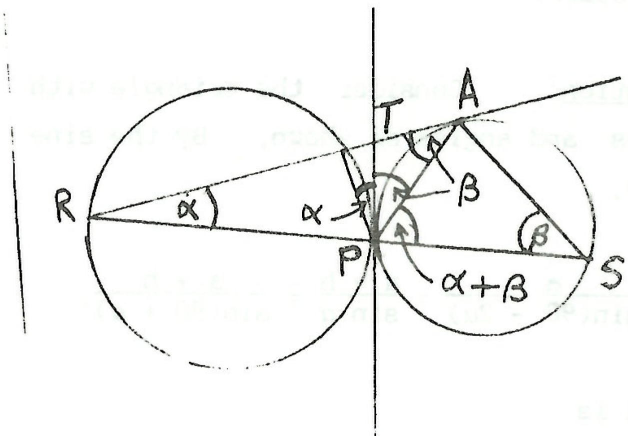
We can find the ratio $\frac{h}{a}$ from

$$1 = \sin^2 \alpha + \cos^2 \alpha = \frac{4h^2(a-h)^2}{(a^2+h^2)^2} + \frac{4h^2(a+h)^2}{(a^2+h^2)^2} = \frac{8h^2(a^2+h^2)}{(a^2+h^2)^2}.$$

Thus $a^2 + h^2 = 8h^2$, that is $h = \frac{1}{\sqrt{7}}$. So the sides of the triangle are in the ratio

$1 - \frac{1}{\sqrt{7}} : 1 : 1 + \frac{1}{\sqrt{7}}$.

6. Two circles are tangent at a point P and A is a point on one of the circles. The tangent to this circle at A intersects the other circle at the points Q and R. Show that A is equidistant from the lines PQ and PR.

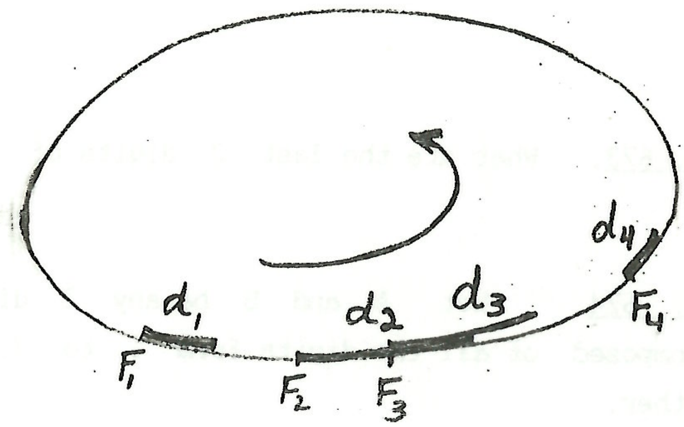


Solution: Draw the common tangent to the two circles PT. The common tangent cuts AQ at T. We need the fact that the angle between a tangent to a circle and a chord through the point of contact is equal to the angle subtended by the chord at the circumference. So, using the tangent PT and the chord PQ, $\angle TPQ = \angle PRQ = \alpha$, and using the tangent PT and the chord PA, $\angle TPA = \angle PSA = \beta$. Also, the tangents TP, TA from T to the right hand circle are equal, so ΔAPT is isosceles and $\angle TPA = \angle TAP = \beta$. Next, $\angle APS$ is an exterior angle of ΔAPR, so $\angle APS = \alpha + \beta$. So the distance from A to PQ is AP $\sin \angle APQ$ = AP $\sin(\alpha + \beta)$ and the distance from A to RS is AP $\sin \angle APS$ = AP $\sin(\alpha + \beta)$. That is A is equidistant from PQ and PR.

7. It takes L litres of fuel to go once round a uniform racing circuit. Badger distributes various amounts of fuel totalling L litres at different locations round the circuit. Show that Toad can find a point on the circuit from which he can get all the way around after starting with an empty tank and using the fuel that he passes on the way. (Toad knows where all the fuel has been placed.)

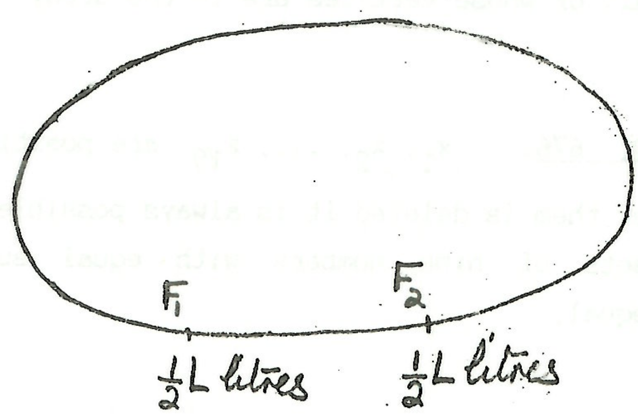
Can Toad always find a starting position from which he can perform this feat in both directions round the circuit?

Solution: If Badger puts all the fuel at one point on the track, Toad can start there, fill up and drive round the track. We will show that the circuit can still be made when Badger distributes the fuel at several points on the track by using induction on the number of fuel dumps. Suppose Badger makes N fuel dumps. Label these $F_1, \ldots, F_N$ in order round the track and at each of these points mark off the distance d_j that Toad could travel from F_j using the fuel dumped there.



Since we have L litres of fuel altogether, exactly enough for one complete circuit, $d_1 + d_2 + \ldots + d_N$ is equal to the total length of the track. If the intervals we have drawn abut neatly end to end, Toad can start at any F_j and drive round the track filling up from each fuel dump as he gets to it, having just exhausted the fuel from the last one. If there are gaps between the intervals, then there must also be overlaps as shown with d_2 and d_3 in the figure. Suppose Toad has reached F_2. Then he can travel a distance $d_2 + d_3$ with the fuel he picks up at F_2 and F_3. He could do exactly the same if all the fuel at F_2 and F_3 was placed at F_2. So we can replace the intervals d_2 and d_3 at F_2 and F_3 by an interval $d_2 + d_3$ at F_2. We now have a problem with $N - 1$ fuel dumps and, by our induction hypothesis, we can find a starting point for this problem which will enable Toad to complete the circuit. The same point will work for the original problem with N fuel dumps.

It may not be possible to find a point that enables Toad to go round the circuit in both directions. For example, suppose half the fuel is distributed at F_1 and half at F_2 which are points quite close together on the track. To go round anticlockwise, Toad must start at F_1: to go clockwise, he must start at F_2.



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