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SOLUTIONS TO PROBLEMS FROM VOLUME 21, NUMBER 3

Q. 648. Prove that no perfect square except 0 is the product of 6 consecutive integers.

Solution: Denote the integers in increasing order by $a_1, a_2, \ldots, a_6$, their product by P. Suppose $P = k^2 > 0$. Then any prime factor p occurs an even number of times when p is expressed as a product of primes. If $p > 6$ it cannot occur as a factor in more than one of the a_i and it follows that each of the integers a_i is of the form $2^{e_i} 3^{f_i} 5^{g_i} t^2$ where t_i is odd. We will show that however the factors 2, 3 and 5 are distributed it is unavoidable that two of the a_i's are perfect squares. This is impossible in 6 consecutive integers if $a_1 > 4$, and it can be checked that P is not a perfect square for $a_1 = 1, 2, 3$ or 4, so that the desired result will then be proved. We shall need to make use of the following facts, nearly all simple and well known.

a) Every second integer is even; every second even integer is a multiple of 4; every 4th even integer is a multiple of 8.

b) Every third integer is a multiple of 3; every fifth integer is a multiple of 5.

c) An odd perfect square exceeds a multiple of 8 by 1 ($(2n+1)^2 = 8[n(n+1)/2] + 1$). Hence if an odd number of the form $3^f 5^g t^2$ exceeds a multiple of 8 by 3 we must have f odd, g even. One can absorb even powers of 3 and/or 5 into t^2, and need only consider f, g equal to 0 or 1. Then the product $3^f 5^g t^2$ exceeds a multiple of 8 by 1, 3, 5 or 7 for the cases $f = g = 0$; $f = 1, g = 0$; $f = 0, g = 1$; $f = 1, g = 1$. Check this.

Since the powers of 2 in the three even integers a_i cannot all have even exponents, two of them must have odd exponents, the other even, so that the sum of the three exponents of 2 is even.

Case 1. The smallest and hence also the largest of the even integers is twice an odd number.

The middle even integer could then be either 4 times an odd number (Case 1 A), or 2^{2m} times an odd number where $m > 1$ (Case 1 B).

Case 1 A. In this case the middle even integer, exceeds a multiple of 8 by 4. The next integer up, since it exceeds a multiple of 8 by 5, contains an odd power of 5 in its prime decomposition (by c above), and whether it is a_4 or a_5 there is no other a_i which is a multiple of 5, so P contains an odd power of 5 and is not a perfect square.

Case 1 B. In this case the middle even integer is a multiple of 8. The next integer down, since it exceeds a multiple of 8 by 7 contains an odd power of 5, and (whether it is a_2 or a_3) no other a_i is a multiple of 5: and P is not a square.

Case 2. The middle even number is twice an odd number. Then one of the others is 4 times an odd number, and the remaining even number is a multiple of 8. If a_5 or a_6 is a multiple of 8, then as before a_4 or a_5 (respectively) must contain an odd power of 5 and P is not a perfect square. This leaves the following possibilities to be considered: a_1 is a multiple of 8 (Case 2A) or a_2 is a multiple of 8 (Case 2B).

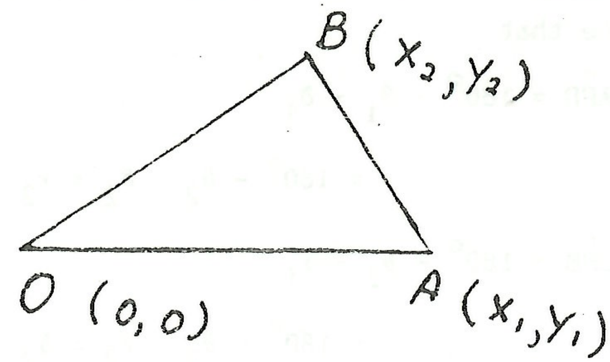
Case 2A. a_4 exceeds a_1, a multiple of 8, by 3, and a_6 exceeds it by 5. Therefore using c, $a_4 = 3\lambda^2$, and $a_6 = 5\mu^2$. Hence neither a_2 nor a_5 contain either 3 or 5 as a factor. Thus a_2 must be an odd perfect square, and a_5 is 4 times an odd square, so is an even square.

Case 2B. a_1 is one less than a multiple of 8, and a_5 is three more than a multiple of 8. By c they must both contain an odd power of 3. This is impossible since they differ by 4.

Thus case 2A is the only remaining possibility. However it results in two of the a_i's being squares, which has already been shown impossible. Thus P cannot be a perfect square unless some factor is 0.

Q. 649. O is a point inside a convex n-gon $P_1P_2 \ldots P_n$ of area A such that $OP_1^2 + OP_2^2 + \ldots \cdot OP_n^2 = 2A$. Prove that the n-gon is a square, and O is the intersection of the diagonals.

Solution: The area of the triangle OAB with vertices $(0,0)$, (x_1,y_1) and (x_2,y_2) is known to be $\frac{1}{2}(x_1y_2 - x_2y_1)$ if the points are in anticlockwise order round the triangle as listed. Therefore



$$OA^2 + OB^2 - 4\text{ Area} = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2x_1y_2 + 2x_2y_1$$

$$= (x_1 - y_2)^2 + (x_2 + y_1)^2 \geq 0$$

The equal sign applies iff $x_1 = y_2$ and $x_2 = -y_1$ (whence $x_1^2 + y_1^2 = x_2^2 + y_2^2$ and $\frac{y_1}{x_1} \cdot \frac{y_2}{x_2} = -1$) i.e. iff OAB is an isosceles triangle with a right angle at O. Now

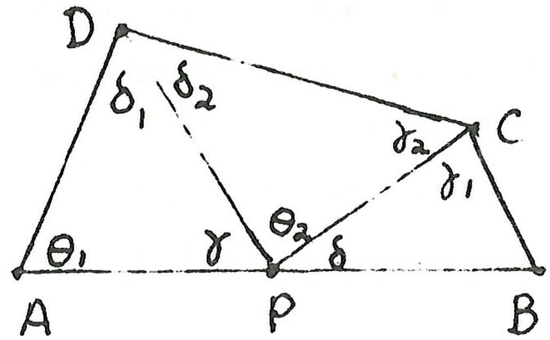
$$OP_1^2 + \ldots + OP_n^2 = \tfrac{1}{2}(OP_1^2 + OP_2^2) + \tfrac{1}{2}(OP_2^2 + OP_3^2) + \ldots + \tfrac{1}{2}(OP_n^2 + OP_1^2)$$

$$\geq 2\text{ Area } OP_1P_2 + 2\text{ Area } OP_2P_3 + \ldots + 2\text{ Area } OP_nP_1.$$

and we have strict inequality unless all lengths OP_i are equal, and all angles $P_i\hat{O}P_{i+1}$ and $P_n\hat{O}P_1$ are right angles. But it is given that equality does obtain. The stated result now follows easily.

Q. 650. The bisectors of the angles C and D of a convex quadrilateral ABCD meet at a point P on AB such that $C\hat{P}D = D\hat{A}B$. Prove that P is the mid point of AB.

Solution: From the figure we easily deduce that



$$\gamma = A\hat{P}D = 180^\circ - \theta_1 - \delta_1$$
$$= 180^\circ - \theta_2 - \delta_2 = \gamma_2$$
$$\delta = C\hat{P}B = 180^\circ - \theta_2 - \gamma_1$$
$$= 180^\circ - \theta_2 - \gamma_2 = \delta_2$$

Therefore $\Delta ADP \sim \Delta PDC \sim \Delta BPC$.

Therefore $\frac{AP}{PD} = \frac{PC}{CD}$ and $\frac{BP}{PC} = \frac{PD}{DC}$.

Therefore $AP = \frac{PC.PD}{CD} = BP$.

Q. 651. Given distances $a_1, a_2, \ldots, a_n$ satisfying

$$a_i < \frac{1}{2}(a_1 + a_2 + \ldots + a_n) \quad \text{for } i = 1, 2, \ldots, n$$

prove that there exists a closed plane polygon with these sides (in the given order).

Solution: Lemma:- Consider first a triangle ABD of which the sides AD and DB are formed from rods AD, DC and CB jointed at D and C and pivoted at the fixed points A, B. It is evident that if the joint C is pushed in the direction of the arrow the triangle can be deformed continuously in its plane, C moving outwards to C′ along the arc of a circle centre B by an arbitrarily small distance, D being pulled to D′, slightly closer to B, along the arc of a circle centre A. Thus a convex quadrilateral is obtained and because of the continuity one can assume that the angles $B\hat{A}D$ and $A\hat{B}C$ have both changed by arbitrarily small amounts.

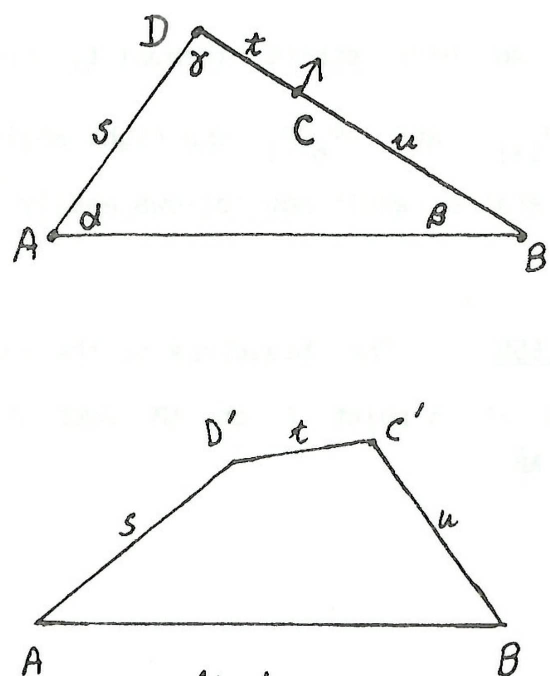


fig 1.

The smallest value of n for which an n-gon exists is 3. In this case, the given condition is equivalent to the assertion that each a_i is less than the sum of the other two. It is a simple and familiar result that a triangle is constructible if and only if the lengths of its sides are so related.

For $n = 4$ let b be the shortest of the combined lengths $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_4$, and consider the three lengths remaining when these two are combined, which we label b_1, b_2, b_3. (For example, if $b_2 = b$, we have $a_1 = b_1$, $a_2 + a_3 = b_2$, $a_4 = b_3$; note that the order of the lengths is still the same as before.) We consider two cases.

Case (i). b is less than the sum of the two remaining lengths. Then it is trivial that $b_i < \frac{1}{2}\sum b_i$ for $i = 1, 2, 3$ whence there is a triangle with sides b_1, b_2, b_3. Using the lemma, this triangle can be deformed into a convex quadrilateral by slightly bending the "combined" aide at the join.

Case (ii). b is equal to the sum of the remaining lengths. The triangle in case (i) collapses to a straight line. By bending the base, b, at the join, its extremities become slightly closer together, and then another triangle can be constructed from the remaining two sides using this distance as the base. Thus a convex quadrilateral is again obtained.

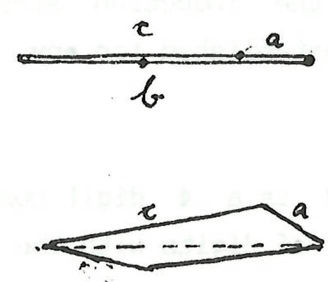


fig 2

We now proceed by induction on n. Suppose we have already established that any set of k (> 4) lengths $a_1, \ldots, a_k$ for which $a_i < \frac{1}{2}\sum_{j=1}^{k} a_j$ can be the sides of a convex k-gon, and consider any set of k + 1 lengths

$$a_1, a_2, \ldots, a_k, a_{k+1} \quad \text{such that} \quad a_i < \frac{1}{2}\sum_{j=1}^{k+1} a_j.$$

Let b be the shortest of the lengths $a_i + a_{i+1}$: $i = 1, 2, \ldots k$. Combine these two lengths into a single length leaving k lengths $b_1, b_2, \ldots, b_k$. This time for <u>any</u> b_i (including whichever one is equal to b) it is immediately evident that $b_i < \frac{1}{2} \sum_{j=1}^{k} b_j$. By our induction assumption, we can construct a convex quadrilateral with these k sides. Consider the side of length b, and an adjacent side b_t (figure 3) in this k-gon. By the lemma the triangle ABD can be deformed into a convex quadrilateral, changing the directions of AD and BC by amounts too small to affect the convexity of the polygon at the vertices B or D. Thus a convex $(k + 1)$-gon can be constructed from the given $(k + 1)$ lengths. This completes the induction step, and the result is established for any n.

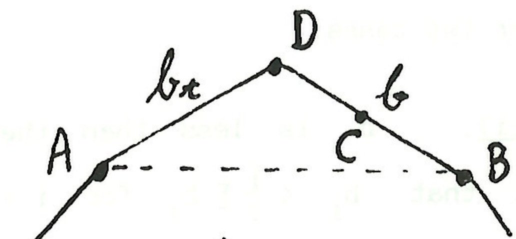


fig 3.

<u>Q. 652</u>. N is a 4 digit number, and M is obtained from N by interchanging the first pair of digits with the last pair. (e.g. if N = 9726, M = 2697). Find N given that $N^2 - M^2$ is a perfect square.

<u>Solution</u>: Let the two digit number formed from the first 2 digits of N be x, and that formed from the last 2 digit be y i.e.

$$N = 100x + y \quad \text{and} \quad M = 100y + x$$

Then $N^2 - M^2 = 9999(x^2 - y^2) = 3^2 \times 11 \times 101 \times (x - y)(x + y)$. This is a perfect square if and only if

$$11 \times 101 \times (x - y)(x + y) \text{ is a perfect square.}$$

i.e.

$$(x - y)(x + y) = 11 \times 101 \lambda^2. \qquad (*)$$

Since $|x - y| < 100$, $x - y$ cannot be a multiple of 101. Since $x + y < 200$ and $x + y$ must be a multiple of 101 we must have, form (*)

$$x + y = 101$$

$$x - y = 11\lambda^2$$

$$x = \frac{101 + 11\lambda^2}{2} \quad y = \frac{101 + 11\lambda^2}{2}$$

The only integer value of λ yielding integers x and y less than 100 is $\lambda = 1$, $x = 56$, $y = 45$. Hence N can only be 5645.

Correct solution from L.A. Koe (James Ruse Agricultural High School).

<u>Q. 653</u>. The sum of two positive integers is 10000000000. The digits of the two numbers are the same but not in the same order. Prove that both numbers are divisible by 5.

<u>Solution</u>: Let $x + y = 10^{10}$. If the final digits of x and y are equal they are 0, or 5 and we are finished. Otherwise let them be a, and $10 - a$ respectively. As the addition proceeds, at every stage the total 10 is obtained, a 1 being carried. i.e. $x_i + y_i = 9$ for every pair of digits in corresponding places in x and y except the units places. If there are m a's in y they are "opposite" m b's (where $b = 9 - a$) in x, and y must also contain m b's in some positions. The corresponding digits in x are a further m a's so that x contains $(m + 1)$ a's when the units digit is included. But x cannot contain more a's then y from the data. Hence it is impossible that the final digits are not equal.

Correct solution from L-A. Koe (James Ruse Agricultural High School).

<u>Q. 654</u>. A class of n students sit for 2 examination papers, the marks being added to give the final result. An entry (4,7,6) opposite a name in the roll book indicates that the student in question came fourth in the first paper, seventh in the second, and sixth in the whole examination after the marks were added. Note that

(1,1,2) is an impossible entry, and so is (16,17,5) if $n = 20$. Find conditions on a,b,c in order that (a,b,c) is an impossible result.

Solution: If a students' entry is (a,b,c) then $a - 1$ students did better in paper 1, and $b - 1$ in paper 2. At most $(a - 1) + (b - 1)$ students did better in at least one paper, and hence the number of students $(c - 1)$ who did better over all cannot exceed $a - 1 + b - 1$. Therefore

$$c - 1 \le a - 1 + b - 1 \qquad c \le a + b - 1.$$

Again $n - a$ students did worse in paper 1, and $n - b$ in paper 2. At most $n - a + n - b$ did worse in least one paper, and hence the number of students who did worse over all, $(n - c)$, cannot exceed $n - a + n - b$

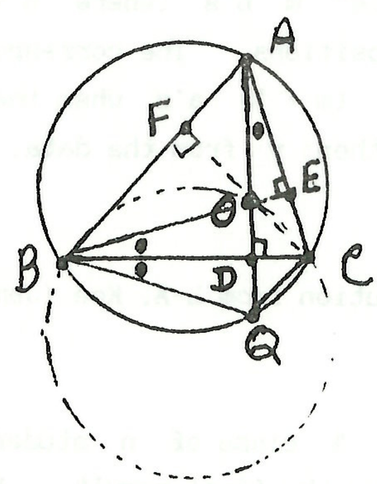
$$n - c \le n - a + n - b \quad \text{therefore} \quad c \ge a + b - n.$$

Hence we conclude that the entry is impossible if either

$$c > a + b - 1 \quad \text{or} \quad c < a + b - n.$$

Q. 655. Given an acute angled triangle ABC, show how to construct a point P such that the circles of the triangles ABC, ABP, BCP and CAP all have equal radii (but do not coincide; i.e. P does not lie on the circumcircle of ΔABC). Prove your construction. Discuss the case when ΔABC is not acute angled.

Solution: Let O be the orthocentre of the triangle ABC, i.e. the intersection of the altitudes. Let AO produced intersect the circumcircle of ΔABC at Q. Then $\Delta BQC \equiv \Delta BOC$ since they share the base BC,



$\hat{QBC} = \hat{QAC}$ (on arc QC)

$= \hat{OBC}$ (from cyclic quadrilateral ABDE),

and similarly

$\hat{QCB} = \hat{OCB}$.

It follows that the circumcircle of ΔBOC is congruent to that of ΔABC. The same is true for the circumcircles of ΔCOA and ΔAOB. Hence P must coincide with the orthocentre of ΔABC, which is easily constructed.

If one interchanges the points A and O, the same figure can be used to establish a similar result for obtuse angled triangles. However if the triangle is right angled at A, the orthocentre coincides with A, and the four circles coalesce into a single circle.

<u>Q. 656</u>. Let E be a set containing n elements and let $A_1, A_2, \ldots, A_r$ be distinct non-empty subsets of E such that

$$A_i \quad A_j \neq \phi, \quad i \neq j \qquad (1)$$

a) Show that $r \leq 2^{n-1}$.

b) If B is an arbitrary subset of E, show that either $A_i \quad B \neq \phi$ for $i = 1, 2, \ldots, r$ or $A_i \quad \bar{B} \neq \phi$ for $i = 1, 2, \ldots, r$, where $\bar{B}$ is the complement of B in E.

c) Show that if $r < 2^{n-1}$ any given collection of subsets $A_1, \ldots, A_r$ satisfying (1) can be extended by finding subsets $A_{r+1}, \ldots, A_{2^n-1}$, so that the collection of $2^n - 1$ subsets still have property (1).

[ϕ denotes the empty set.]

<u>Solution</u>: I regret the two small typing errors corrected in the above) which must have made the question meaningless to all except psychics.

Observe that there are altogether 2^n different subsets of E: to construct a subset S, take each of the n elements in turn and make one of 2 choices put it in E or leave it out - there are $2 \times 2 \times \ldots \times 2$ (n times) different constructions.

a) Let B be any subset of E; and $\bar{B}$ its complement. Then since B $\bar{B} = \phi$ they cannot both be included amongst the A's. Hence r is at most half of the number of all possible subsets, $r \leq \frac{1}{2} \times 2^n = 2^{n-1}$.

b) Suppose it is false that $A_i \cap B \neq \phi$ for every i, i.e. for some i_0, $A_{i_0} \cap B = \phi$. Then since B contains no element of A_{i_0}, $\bar{B}$ must contain every element of A_{i_0}. Therefore $\bar{B} \cap A_j$ includes all elements of $A_{i_0} \cap A_j$ for every j, and this is not ϕ by (1). The result is now clear.

c) If $r < 2^{n-1}$ there must be at least one subset B such that neither B nor $\bar{B}$ is included amongst $A_1, A_2, \ldots, A_r$. By (b) one or other of these sets can now be labelled A_{r+1} while still preserving property (1). This process can be continued until we have 2^{n-1} subsets A_i.

<u>Q. 657</u>. We have 30 locked boxes and 30 keys, each opening just one lock. Each box has an opening through which we throw one key at random.

i) We break one of the boxes. What is the probability that we can open all the other boxes without breaking them?

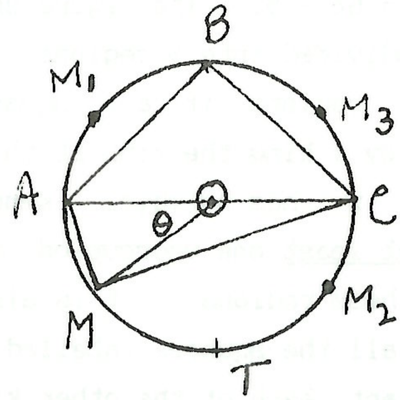
ii) If we initially break two boxes instead of one, what is the probability that we can open all the other boxes without breaking them?

<u>Solution</u>: There are 30! different ways altogether of placing the 30 keys into the boxes. When one box x is broken all the others can be opened if the boxes can be placed in a "cycle" $(x, b_1, b_2, \ldots, b_{29})$ such that the key in x opens b_1, and the key in b_i opens b_{i+1} for $i = 1, 2, \ldots, 28$.
[Then the key in b_{29} must belong to x.] Since there are 29 unbroken boxes, there are 29! different cycles corresponding to all possible arrangements of these box numbers as $b_1 \ldots b_{29}$. Hence 29! of the 30! ways of distributing the keys permit all boxes to be opened. The probability of this is $\frac{29!}{30!} = \frac{1}{30}$.

ii) In addition to the above 29! distributions, the boxes can all be opened if the broken boxes x and y are in two different cycles which include all the boxes between them. e.g. a figure $(x, b_1, b_2, \ldots, b_k)\ (y, b_{k+1}, \ldots, b_{28})$ would describe such a situation, where the key in x opens box b_1, the key in b_k belongs to x; the key in y opens b_{k+1}, the key in b_{28} belongs to y, and the key in b_i opens b_{i+1} for all values i from 1 up to 27, except k. (This wording would require slight modification in two extreme cases: $k = 0$, x contains its own key; and $k = 28$, y contains its own key.) We can construct figures like the above by starting with (x,) and filling the gap with any arrangement of the 29 box numbers, other then x. (The figure is then completed by placing)(to the left of y.) Again there are 29! different such figures. Thus, altogether there are 29! + 29! distributions of the keys which permit the boxes to be opened and the probability is now $\frac{29! + 29!}{30!} = \frac{2}{30} = \frac{1}{15}$.

Correct solution: L-A. Koe supplied a correct working of (i).

Q. 658. The triangle ABC is isosceles and has a right angle at B. The point M is on the circumcircle of ABC. Find all positions of M such that it is possible to construct a triangle from the segments BC, MA, MC.



Solution: It will be sufficient to consider M on the quadrant AT of the circumcircle in the figure, since for the symmetrically located points M_1, M_2 and M_3, we obtain line segments of the same lengths as MA and MC. Let $\hat{MOA} = \theta$ where O is the centre of the circumcirle, which coincides with the mid point of AC. If R is the radius of the circle one obtains $2R \sin\frac{\theta}{2}$, $2R \cos\frac{\theta}{2}$, $\sqrt{2}R$ for the lengths of MA, MC, and AB respectively. A triangle can be constructed if the longest length MC is less than the sum of the others:-

$$2R \cos\frac{\theta}{2} < 2R \sin\frac{\theta}{2} + \sqrt{2}R$$

i.e.

$$\frac{1}{\sqrt{2}} \cos\frac{\theta}{2} - \frac{1}{\sqrt{2}} \sin\frac{\theta}{2} < \frac{1}{2}.$$

$$\cos(\frac{\theta}{2} + \frac{\pi}{4}) < \frac{1}{2}$$

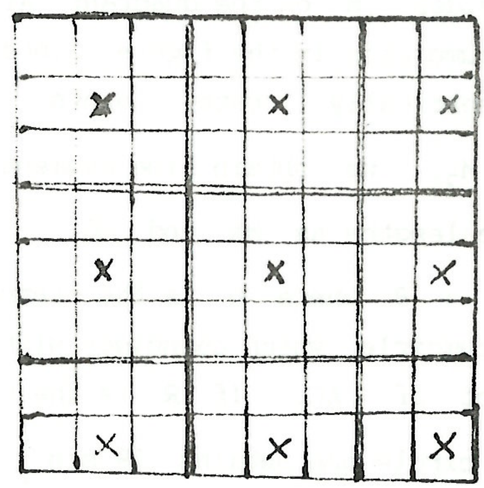
$$\frac{\theta}{2} + \frac{\pi}{4} > \frac{\pi}{3}.$$

$$\frac{\theta}{2} > \frac{\pi}{12}. \quad \theta > \frac{\pi}{6}.$$

Hence a triangle can be constructed provided $M\hat{O}A$ and $M\hat{O}C$ exceed 30°.

Q. 659. In chess a king can move to any adjacent unoccupied square, horizontally, vertically, or diagonally. What is the maximum number of kings that can be placed on an 8×8 chessboard so that each king has at least one square to which it can move. (For example, on a 2×2 board, the answer would be three.) Justify your answer.

Solution: The maximum number of kings is $55 = 64 - 9$. The figure shows the board subdivided into 9 regions. In any of those regions if all squares are occupied by a king the king at the square labelled x has no move. Hence there must be at least one unoccupied square in each of these regions. It is also clear that if all the squares labelled x are left vacant each of the other kings can move to one of these squares.



Correct solution from L-A. Koe (James Ruse Agricultural High School).

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