MOESSNER–LONG SIEVE PATTERNS

by

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Abstract.

We start with the sieve of Eratosthenes for generating primes, and then look at other ways of sieving the integers. We get some curious sequences of numbers and some elegant relations.

Introduction.

The oldest number "sieve" goes back to about 230 BC when Eratosthenes showed how to find the primes. Start with a list of the integers beginning with 2. Leave 2, but strike out every second number, 4, 6, 8 and so on. The first number not struck out is 3. Leave 3, but strike out every third number, 6, 9, 12, and so on. Continue in this way. The numbers which remain are the primes (see Table 1).

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~ ...

Table 1.

Recently much research has gone into sieve methods. The sieves we shall consider were discovered by Alfred Moessner in 1951 and have since been generalized by Calvin Long.

Some Examples.

This time we list the positive integers, strike out every even integer and look at the remaining partial sums:

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Number	Partial Sums	= Total
1	1	= 1
2		
3	1 + 3	= 4
4		
5	1 + 3 + 5	= 9
6		
7	1 + 3 + 5 + 7	= 16
8		
9	1 + 3 + 5 + 7 + 9	= 25
10		
11	1 + 3 + 5 + 7 + 9 + 11	= 36
12		
13	1 + 3 + 5 + 7 + 9 + 11 + 13	= 49
14		
15	1 + 3 + 5 + 7 + 9 + 11 + 13 + 15	= 64
16		
17	1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17	= 81
18		
19	1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19	= 100
20		

Table 2

The sequence of partial sums is obviously the set of squares. This can be expressed by

$$n^2 = \sum_{j=1}^{n} (2j - 1)$$

which you can prove by mathematical induction.

Next suppose we start with the positive integers and delete every third integer form partial sums again, then delete every second term of this new sequence as in Table 3, and form new partial sums:

1	2	~~3~~	4	5	~~6~~	7	8	~~9~~	10	11	~~12~~	13	14	~~15~~	16	17	~~18~~	...
1	~~3~~		7	~~12~~		19	~~27~~		37	~~48~~		61	~~75~~		91	~~108~~		...
1			8			27			64			125			216			... .

Table 3.

Now we have the sequence of cubes: $n^3 = \sum_{j=1}^{n} (3j(j-1)+1)$.

Moessner conjectured that the process could be generalized, and it can, as a little investigation will reveal. The sequence of kth powers can be generated by striking out every kth integer from the sequence of positive integers, forming the partial sums, deleting every $(k-1)$th number, forming partial sums, deleting every $(k-2)$th number, and so on, through $k-1$ stages. This was proved by Oskar Perron in 1951 using mathematical induction. Try it out yourself for $k = 4$. There are also various combinatorial relations which arise in the formation but investigating those is beyond the concern of this article.

<u>General Cases</u>.

In 1966, Long used an interesting generalization of Pascal's triangle and mathematical induction to prove that if Moessner's process is applied to the arithmetic sequence

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \ldots,$$

where a, d are positive integers, and we start by deleting every kth integer then the final series of partial sums is

$$a.1^{k-1}, \quad (a+d).2^{k-1}, \quad (a+2d).3^{k-1}, \quad \ldots .$$

We see this for $k = 3$ in Table 4

a	a + d	~~a + 2d~~	a + 3d	a + 4d	~~a + 5d~~	a + 6d	...
a	~~2a + d~~		3a + 4d	~~4a + 8d~~		5a + 14d	...
a			4a + 4d			9a + 18d	...

Table 4.

Of course, when $a = d = 1$, and $k = 2$ we get the squares, as in Table 2, and when $a = d = 1$ and $k = 3$, we get the cubes, as in Table 3.

We notice that the deletions move one column to the left as we go down the rows. We next try this approach for other patterns of numbers.

~~1~~	2	~~3~~	4	5	~~6~~	7	8	9	~~10~~	11	12	13	14	~~15~~	16	...
	2		6	~~11~~		18	26	~~35~~		46	58	71	~~85~~		101	...
			6			24	~~50~~			96	154	~~225~~			326	...
						24				120	~~274~~				600	...
										120					720	...

Table 5.

We start by deleting the triangular numbers 1, 3(= 1 + 2), 6(= 1 + 2 + 3), 10(= 1 + 2 + 3 + 4), ..., form partial sums, delete the numbers one column to the left, form partial sums, and so on, and eventually the numbers that remain are the factorials, 1, 2(= 1 × 2), 6(= 1 × 2 × 3), 24(= 1 × 2 × 3 × 4), ...

More generally, if $k_1, k_2, k_3, \ldots,$ is any sequence of positive integers, and we start by deleting the numbers

$$u_1 = k_1, \quad u_2 = k_2 + 2k_1, \quad u_3 = k_3 + 2k_2 + 3k_1, \quad u_4 = k_4 + 2k_3 + 3k_2 + 4k_1,$$

and so on, and apply the process described above, the numbers that remain are

$$v_1 = 1^{k_1}, \quad v_2 = 1^{k_2} \times 2^{k_1} \quad v_3 = 1^{k_3} \times 2^{k_2} \times 3^{k_1}, \quad v_4 = 1^{k_4} \times 2^{k_3} \times 3^{k_2} \times 4^{k_1},$$

and so on. (Table 5 is given by $k_1 = k_2 = k_3 = \ldots = 1$.)

We can use this general result to investigate sieves associated with other sequences. Moreover, the Moessner-Long process can be generalized still further. For instance, if we start by deleting the (u_n) and (u'_n), and the corresponding obtained sequences are (v_n) and (v'_n), then starting by

$$\text{deleting } (u_n + u'_n) \text{ yields } (v \times v'_n)$$

and

$$\text{deleting } (u_n - u'_n) \text{ yields } (v_n \div v'_n).$$

The proofs are quite challenging but within the scope of an interested and tenacious school student. The work lends itself to computer analysis and involves pattern recognition and induction.

References.

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