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In this issue we first look at two problems, set in 1985, which use mathematical induction. Firstly, from the 3 unit paper,

Problem 86/4. Use the principle of Mathematical Induction to prove that $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n.

Solution: Students should be familiar with the mathematical induction steps, which are:

Step 1. Assume the proposition $P(k)$ is true for an integer k.

2. Prove that $P(k + 1)$ can be deduced from $P(k)$.

3. Prove that $P(1)$ is true.

4. Then $P(n)$ follows from steps 1, 2, and 3.

In this problem the steps are as follows:

1. Assume that for a positive integer k, $5^k + 2(11^k) = 3N$ where N is some integer. It follows that $5^k = 3N - 2(11^k)$.

2. Now consider

$$\begin{aligned} 5^{k+1} + 2(11^{k+1}) &= 5(5^k) + 22(11^k) \\ &= 5[3N - 2(11^k)] + 22(11^k) \\ &= 15N + 12(11^k) = 3[5N + 4(11^k)] \end{aligned}$$

Thus, if $5^k + 2(11^k)$ is a multiple of 3, then $5^{k+1} + 2(11^{k+1})$ is also a multiple of 3.

3. For $n = 1$, $5^n + 2(11^n) = 5 + 22 = 27$, which is a multiple of 3. Thus, by induction, $5^n + 2(11^n)$ is a multiple of 3.

Problem 86/5. From the 4 unit paper.

a) Show that for $k > 0$,

$$2k + 3 > 2\sqrt{((k + 1)(k + 2))}.$$

b) Hence prove that for $n > 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2[\sqrt{(n + 1)} - 1].$$

c) Is the statement that, for all positive integers N,

$$\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10}$$

true? Give reasons for your answer.

<u>Solution</u>. a) Let $A = 2k + 3$, $B = 2\sqrt{(k+1)(k+2)}$, and, since $k > 0$, $A > 0$, $B > 0$. Then

$$\begin{aligned} A^2 - B^2 &= (2k+3)^2 - 4(k+1)(k+2) \\ &= 4k^2 + 12k + 9 - 4k^2 - 12k - 8 \\ &= 1. \end{aligned}$$

Hence $A^2 > B^2$, and since $B > 0$, $A > 0$, it follows that $A > B$.

b) Let $S(k) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}}$. <u>Assume</u> $S(k) > 2[\sqrt{(k+1)} - 1]$.

Then $S(k+1) = S(k) + \frac{1}{\sqrt{(k+1)}}$

$$> 2[\sqrt{(k+1)} - 1] + \frac{1}{\sqrt{(k+1)}} = \frac{2k + 3 - 2\sqrt{(k+1)}}{\sqrt{(k+1)}}$$

But from (a), $2k + 3 > 2\sqrt{(k+1)(k+2)}$, and <u>hence</u>

$$S(k+1) > \frac{2\sqrt{(k+1)(k+2)} - 2\sqrt{(k+1)}}{\sqrt{(k+1)}} = 2(\sqrt{(k+2)} - 1)$$

after cancelling $\sqrt{(k+1)}$, which is a common factor in the numerator.

But $S(1) = 1$, and, for $k = 1$, $2\sqrt{(k+1)} - 2 = 2(\sqrt{2} - 1) > 1$. Hence

$$S(n) > 2[\sqrt{(n+1)} - 1]$$

c) No. For example, simply let $N = 10^{30} - 1$. Then

$$S(N) > 2\sqrt{10^{30}} - 2 = 2(10^{15}) - 2 > 10^{10}!$$

Note that the result in (b) is that $S(n)$ becomes infinitely large as $n \to \infty$, thus proving that the series diverges.

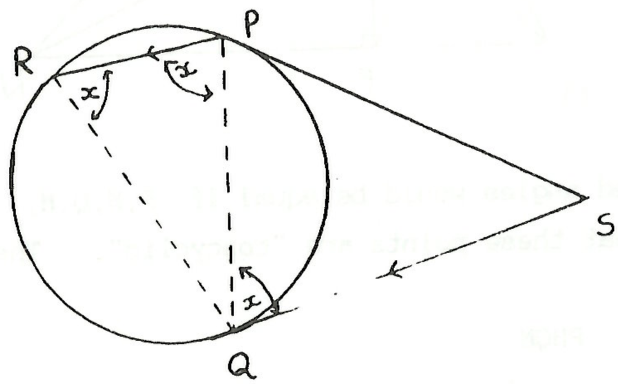
This issue we also have space to examine the two geometry problems. From the 3 unit paper.

Problem 86/6. P, Q are points on a circle and the tangents to the circle at P, Q meet at S. R is a point on the circle so that the chord PR is parallel to QS.

a) Draw a neat sketch in your answer book, showing the given information.

b) Giving reasons, prove carefully that QP = QR.

Solution. The key to any geometry problem is a good diagram, covering an area of at least 6cm × 8cm of your answer book. Your diagram should look something like thus:



Join the points RQ, QP as in the figure, then, if we are to prove QP = QR, we need to prove ΔPQR is isosceles. The proof should be written out as follows:

RP//QS (given) therefore ∠RPQ = ∠PQS (Corresponding angles).

But SQ is a tangent at Q (given)

therefore ∠PQS = ∠PRQ (alternate segment theorem)

therefore ∠RPQ = ∠PRQ

therefore ΔPQR is isosceles

therefore QP = QR

A more difficult question from the 4 unit paper:

Problem 86/7. In an acute-angled triangle with vertices L, M, N, the foot of the perpendicular from L to MN is P, and the foot of the perpendicular from N to LM is Q. The lines LP, QN intersect at H.

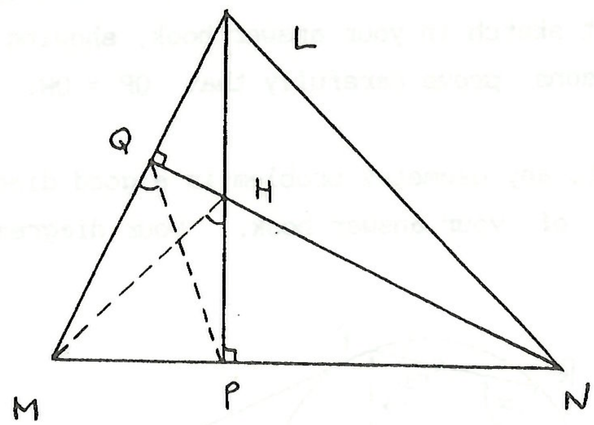
α) Draw a clear diagram showing the given information.

β) Prove that ∠PHM = ∠PQM.

γ) Prove that ∠PHM = ∠LNM.

δ) produce MH to meet LN at R. Prove that MR ⊥ LN.

Solution: α)



β) The two required angles would be equal if P,H,Q,M, all lie on a circle - so we need to prove that these points are "concyclic". The formal "proof" is as follows.

Consider the quadrilateral PHQM

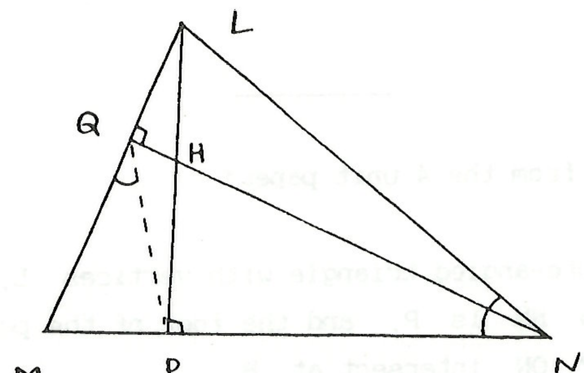
$\angle HPM = 90^\circ$ (given), $\angle HQM = 90^\circ$ (given)

therefore $\angle HPM + \angle HQM = 180^\circ$

therefore PHQM is a cyclic quadrilateral

therefore $\angle MQP = \angle MHP$ (angles in the same segment) (A)

γ) Note that $\angle MQP$ is the external angle of the quadrilateral LNPQ, so we need to prove



that LNPQ is a cyclic quadrilateral. The proof is as follows:

Consider the quadrilateral LNPQ

$\angle L\hat{Q}N = \angle L\hat{P}N = 90^\circ$ (given).

These are angles in the same segment, therefore L, N, P, Q are concyclic.

Therefore $\angle MQP = \angle LNP$ (ext. angle of a cyclic quad.) (B)

Thus, from (A) and (B), it follows that

$\angle PHM = \angle LNM$ (since MPN is a str. line)

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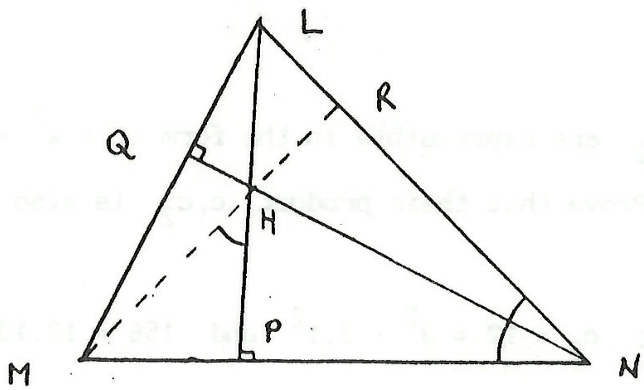
8)



We have proved that $\angle MHP = \angle RNP$. Hence RNPH is a cyclic quadrilateral (ext $\angle$ = int. opp $\angle$).

Hence $\angle HPN + \angle HRN = 180^\circ$ (opp $\angle$s are supplementary)

But $\angle HPN = 90^\circ$ (given)

therefore $\angle HRN = 90^\circ$

therefore $MR \perp LN$

We have actually shown that the three altitudes of an acute angled triangle meet at a single point. The point H is called the "orthocentre" of the triangle.

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