# SOLUTIONS TO PROBLEMS FROM VOLUME 22, NUMBER 1

Q. 660 A jailer walks $n$ times past a row of $n$ cells walking always from left to right. Under the terms of a partial amnesty, the following happens.

Initially all cells are locked.

On the first pass, he turns every lock, beginning with the first.

On the second pass, he turns every second lock, beginning with the second.

And so on.

More succinctly:-

On the $k$th pass he turns every $k$th lock beginning with the $k$th ($k = 1, 2, \ldots, n$).

Which cells are now unlocked? Prove your assertion.

Solution: For purposes of experiment let us consider cell #12. The lock is turned on the $k$th pass where $k = 1, 2, 3, 4, 6$, and 12. i.e. an even number of times. So cell #12 ends up locked. We see that the key of cell #$n$ is turned $d(n)$ times where $d(n)$ denotes the number of distinct positive integers which are factors of $n$ (including 1 and $n$). Cell #$n$ ends up unlocked if and only if $d(n)$ is odd. Now if $r$ is a factor of $n$ less then $\sqrt{n}$ then $\frac{n}{r}$ is also a factor greater than $\sqrt{n}$ (and vice versa). Hence every integer $n$ has the same number of factors less than $\sqrt{n}$ and greater than $\sqrt{n}$. Hence $d(n)$ is even unless $n$ is a perfect square (in which case $\sqrt{n}$ is one additional integer factor). Thus cell #$n$ ends up unlocked for $n = 1, 4, 9, \ldots, k^2$ and for no others.

Correct solution received from Lisa-Ann Koe (James Ruse High School).

Q. 661 Let $P_n(x)$ be the polynomial

$$P_n(x) = 1 + 2x + 3x^2 + \ldots + (n + 1)x^n.$$

Show that $P_n(x)$ has no real zero if $n$ is even, and exactly one real zero if $n$ is odd, lying between $-1$ and $0$.

Solution: $$P_n(x) = 1 + 2x + 3x^2 + \dots + (n+1)x^n \qquad (1)$$

$$xP_n(x) = x + 2x^2 + \dots\dots\dots + nx^n + (n+1)x^{n+1}.$$

Therefore

$$(1 - x)P_n(x) = (1 + x + \dots + x^n) - (n+1)x^{n+1} = \frac{1 - x^{n+1}}{1 - x} - (n+1)x^{n+1}$$

giving

$$P_n(x) = \frac{1 - (n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2} \qquad (2)$$

From (1), it is clear that $P_n(x) > 0$ whenever $x > 0$, so if $P_n(\alpha) = 0$ $\alpha$ must be negative. From (2) it is now clear that $P_n(x) = 0$ if and only if

$$1 - (n+2)x^{n+1} + (n+1)x^{n+2} = 0 \text{ with } x \text{ negative.} \qquad (3)$$

If $n$ is even, all terms on the L.H.S. of (3) are positive (since $x < 0$), and hence $P_n(x)$ has no zeros in this case.

If $n$ is odd, writing $x = -y$, (3) becomes

$$(n+1)y^{n+2} + (n+2)y^{n+1} = 1 \text{ where } y > 0. \qquad (4)$$

Since both terms on the L.H.S. of (4) increase steadily as $y$ increases, and the L.H.S. grows from $0$ at $y = 0$ to $2n + 3$ when $y = 1$, it is clear that equality applies for exactly one value of $y$, and that it lies between 0 and 1. Hence when $n$ is odd $P_n(x)$ has a single zero which lies between $-1$ and $0$.

Q. 662 Let N be the number of strings of 20 letters all either O or X, which contain the two letter word OX precisely 7 times. (e.g. such a string as XXOXOOXOXXXOXOXXOXOX where the occurrences of OX have been underlined). Find arguments to show that

a) $$N = \binom{7}{7}\binom{13}{7} + \binom{8}{7}\binom{12}{7} + \binom{9}{7}\binom{11}{7} + \binom{10}{7}\binom{10}{7} + \binom{11}{7}\binom{9}{7} + \binom{12}{7}\binom{8}{7} + \binom{13}{7}\binom{7}{7}$$

and

b) $$N = \binom{21}{15}$$

(Note: $\binom{n}{r}$ denotes the binomial coefficient for which ${}^nC_r$ is an alternative notation)

Solution: a) Suppose there are altogether k X's and 20 - k O's. Obviously $7 \leqslant k \leqslant 13$. Place the k X's in a row and select the 7 of them which are to form one of the occurrences of "OX". There are $\binom{k}{7}$ ways to do this. Make a gap in the row of X's to the left of each chosen one, to make room for the insertion of O's.

Similarly place the 20 - k O's in a row, choose 7 of them ($\binom{20-k}{7}$ different ways possible) to form the OX's and separate the row at the right of each chosen O. Finally interleave the two sets to form the 7 OX's. It is clear that this process can be performed in $\binom{k}{7} \times \binom{20-k}{7}$ different ways, and that every possible sequence of k X's asnd 20 - k O's containing 7 OX's will be obtained exactly once.

Adding for all possible values of k from 7 to 13 gives the result a).

b) The sequence can begin with an O, or with an X.

i) If it begins with an O there are seven blocks of O's each followed by a block of X's. (There may also be a final block of O's.)

Let $x_1, x_2, \ldots, x_{14}$ be numbers indicating the position of the last symbol in these 14 blocks. e.g. 1, 3, 5, 6, 8, 9, 11, 12, 13, 15, 16, 17, 18, 19 would correspond to the arrangement

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| O | X | X | O | O | X | O | O | X | O | O | X | O | X | X | O | X | O | X | O |

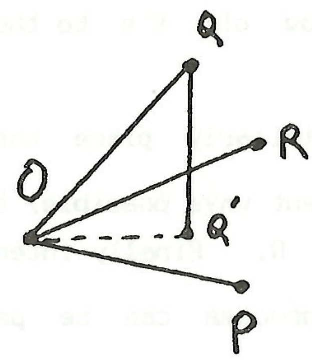
Conversely, every choice of 14 numbers from {1, 2, ..., 20} can be arranged in increasing order and used to determine exactly one such arrangement. Hence there are ${}^{20}C_{14}$ such arrangements.

ii) If there is a preliminary block of X's before the first block of O's, the arrangement can be specified instead by giving the position of the last symbol in each of the first 15 blocks. (There will be also a concluding block of O's if the last number in our list is less than 20.) Since there are ${}^{20}C_{15}$ ways of choosing 15 different numbers from the first 20 positive integers, there will be ${}^{20}C_{15}$ different arrangements of O's and X's of this type.

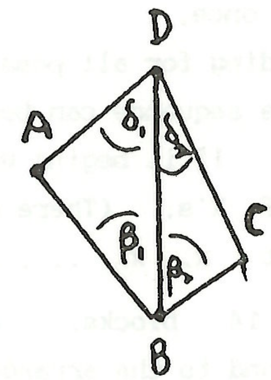
Hence the total number of arrangements is ${}^{20}C_{14} + {}^{20}C_{15} = {}^{21}C_{15}$ (the "Pascal Triangle" property of the binomial coefficients).

Q. 663 ABCD is a skew quadrilateral (i.e. the vertices do not all lie in one plane) such that $\angle ABC = \angle BCD = \angle CDA = 90^{\circ}$. Prove that $\angle DAB$ is acute.

Solution: It is a theorem that for any three concurrent lines OP, OQ, OR not all in one plane, the sum of any two dihedral angles $P\hat{O}Q$, $Q\hat{O}R$, and $P\hat{O}R$, exceeds the third; e.g. $P\hat{O}Q + Q\hat{O}R > P\hat{O}R$. (If $Q'$ is the foot of a perpendicular from Q to the plane POR, it is not difficult to show that $P\hat{O}Q > P\hat{O}Q'$ and $Q\hat{O}R > Q'\hat{O}R$, from which the desired result follows immediately.)




Let ABCD be any skew quadrilateral. Draw the diagonal BD, and denote the angles $A\hat{B}D$, $C\hat{B}D$, $A\hat{D}B$, $C\hat{D}B$ by $\beta_1$, $\beta_2$, $\delta_1$, $\delta_2$ respectively as in the figure. Since the angles in a triangle add to $180^{\circ}$ we have $(\hat{A} + \beta_1 + \delta_1) + (\hat{C} + \beta_2 + \delta_2) = 360^{\circ}$. But by the theorem quoted above $\beta_1 + \beta_2 > A\hat{B}C$ and $\delta_1 + \delta_2 > A\hat{D}C$. It follows that

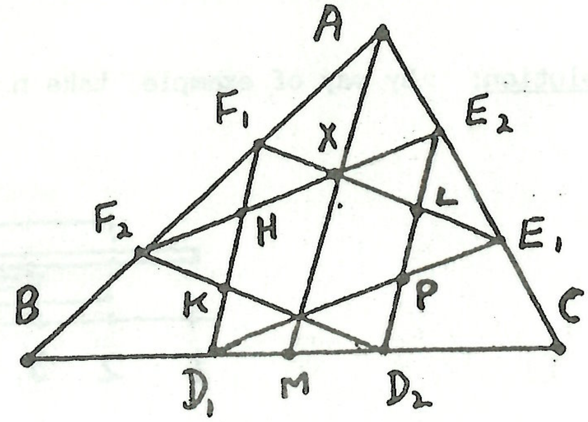



$$360^{\circ} > \hat{A} + A\hat{B}C + \hat{C} + A\hat{D}C.$$

i.e. The angles of any skew quadrilateral add to less than $360^{\circ}$ In particular, if any three of its angles are right angles, the fourth angle must be acute.

Q. 664 Each side of a triangle of area 1 unit is divided into three equal parts. The six points of division are the vertices of two triangles whose intersection is a hexagon. Find the area of the hexagon

Solution: The only possible way to obtain triangles which overlap in a hexagon is as shown in the figure. Since $E_1F_1$ and $E_2F_2$ are medians of $\Delta AF_2E_1$, their point of intersection X trisects each of them. Since $F_2E_1 \parallel BC$ the third median AX produced bisects not only $F_2E_1$, but BC also. (Use similar triangles for a formal proof.) Hence $\frac{BF_1}{BA} = \frac{BD_1}{BM} = \frac{2}{3}$ and it follows that $F_1D_1 \parallel AX$. Therefore H is the midpoint of $F_2X$ (because $F_1$ is the midpoint of $F_2A$).




We have shown that H,X are points of trisection of $E_2F_2$. Similarly, each side of $\Delta D_1E_1F_1$ or of $\Delta D_2E_2F_2$ is trisected by its intersections with the other triangle. We can now see that all 9 small triangles in the figure have equal areas, either because they stand on equal bases (e.g. $\Delta F_2HF_1 = \Delta HXF_1$, because base $F_2H$ = base HX) or because they are congruent (e.g. $\Delta HXF_1 \equiv \Delta E_2XL$). Since

$$\text{area } \Delta AF_2E_2 = \frac{2}{3} \times \frac{1}{3} \times \text{area } \Delta ABC = \frac{2}{9}$$

and

$$\text{area } \Delta F_2F_1X = \frac{1}{2} \times \frac{2}{3} \times \text{area } \Delta F_2AE_2 = \frac{1}{2} \times \frac{2}{3} \times \frac{2}{9} = \frac{2}{27}$$

we see that each of the 9 small triangles has area

$$\frac{1}{2}\ \Delta F_2F_1X = \frac{1}{27}.$$

$$\text{area } AF_1XE_2 = \Delta AF_2E_2 - \Delta F_2F_1X$$

$$= \frac{2}{9} - \frac{2}{27} = \frac{4}{27}.$$

and similarly area $BF_2KD_1$ = area $CE_1PD_2 = \frac{4}{27}$. Hence

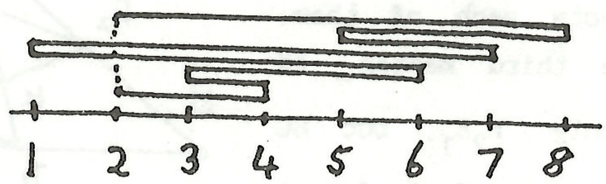
$$\text{area of hexagon} = 1 - 3 \times \frac{4}{27} - 9 \times \frac{1}{27} = \frac{2}{9}.$$

Q. 665 The sequence $(a_1, a_2, \ldots, a_n)$ is a permutation (i.e. rearrangement) of $(1, 2, \ldots, n)$

a) Prove that $|a_1 - a_2| + |a_2 - a_3| + \ldots + |a_{n-1} - a_n| + |a_n - a_1| \geq 2n - 2$.

b) For how many distinct permutations of $(1, 2, \ldots, n)$ does equality hold in (a).

Solution: By way of example, take $n = 8$ and $(a_1, a_2, \ldots, a_8) = (2,4,3,6,1,7,5,8,)$.



Then $S = |a_1-a_2| + |a_2-a_3| + \ldots + |a_n-a_1|$ can be represented as the length of a loop of string constructed as follows (see figure): start at the point 2 on the number axis, lay down in succession straight pieces of string joining 2 to 4, 4 to 3, ..., (and so on as prescribed by the order of numbers in the permutation), finally laying down a piece from 8 to 2. Now join the adjacent ends of the pieces of string. It is clear that S = the total length of string used, and that the string forms a closed loop which extends from the point 1 on the number axis to the point 8. Obviously each section of the number axis between consecutive integers in [1,8] must be "covered" at least twice by the loop of string, so its length must be at least $2 \times (8 - 1)$. Its length will be greater if any such section is covered more than twice e.g. in our figure, the interval (3,4) is covered 6 times by the loop.

a) In general $S = |a_1 - a_2| + \ldots + |a_{n-1} + a_n| + |a_n - a_1|$ is equal to the length of a loop of string extending from 1 to n on the number axis, therefore

$$S \geqslant 2 \times (n - 1)$$

b) $S = 2 \times (n - 1)$ if and only if the loop covers each interval exactly twice.

Note that in our example the same loop of string is obtained if we consider the permutation (1, 7, 5, 8, 2, 4, 3, 6) obtained from the original one by cyclically permuting it (i.e., moving the first number to the end) until it begins with 1. Consider a permutation $1, a_2, a_3, \ldots, a_{j-1}, n, a_{j+1}, \ldots, a_n$ beginning with 1 and having n in the jth place. The corresponding loop of string will cover each part of the axis only twice if and only if $a_2, a_3, \ldots, a_{j-1}$ are in increasing order, and $a_{j+1}, a_{j+2}, \ldots, a_n$ are in decreasing order of magnitude.

The $n - 2$ numbers $2, 3, \ldots, n - 1$ can be partitioned into two sets $S_1$ and $S_2$ in $2^{n-2}$ different ways,* and for each of these there is a unique sequence beginning with 1, then having the numbers in $S_1$ in increasing order followed by n, and then by the numbers in $S_2$ in decreasing order. Each of these $2^{n-2}$ permutations can now be cyclically permuted leaving 1 in any one of the n places. Hence there are $n \times 2^{n-2}$ different permutations of 1, 2, ..., n for which S is equal to $2n - 2$.

(*Proof:- To construct $S_1$, take each of the $n - 2$ numbers in turn and make one of 2 choices: place it in $S_1$, or leave it out. The number of different constructions is thus $2 \times 2 \times \dots \times 2$ where there are $n - 2$ factors.)

Q. 666 Can you find an infinite set of natural numbers, S such that for any subset A of S $\sum_{s \in A} s$ is never a perfect square?

What if "square" is replaced by "cube" in the above, or by kth power, (k being any given integer > 3)?

Solution: Consider the set $S = \{2, 2^3, 2^5, \dots\}$ consisting of all powers of 2 with odd exponents. Then if A is any subset of S with k elements

$$N = \sum_{s \in A} s = 2^{\ell_1} + 2^{\ell_2} + \dots + 2^{\ell_k}$$

(where we may assume $\ell_1 < \ell_2 < \dots < \ell_k$)

$$= 2^{\ell_1}(1 + 2^{\ell_2-\ell_1} + \dots + 2^{\ell_k-\ell_1}).$$

Since $2^{\ell_j-\ell_1}$ is even for $j = 2, 3, \dots, k$ the factor in parentheses is an odd number. Hence the power of 2 contained as a factor in the factorisation of N into primes is $2^{\ell_1}$, having an odd exponent. It follows that N is not a perfect square.

If instead we want N to be not a perfect cube, take for example

$$S = \{2, 2^2, 2^4, 2^5, 2^7, \dots\}$$

consisting of all powers of 2 for which the exponent is not a multiple of 3. Then $N = 2^{\ell_1}$ X odd number, as before, and this cannot be a perfect cube, because $\ell_1$ is not a multiple of 3.

Likewise, if S consists of all powers of 2 having exponent not a multiple of k, one similarly sees that n is not a perfect kth power.

Q. 667 Let $f(k)$ denote the number of zeros in the decimal representation of the natural number $k$. Compute $S_n = \sum_{k=1}^{n} 2^{f(k)}$ where $n = 10^{10} - 1$.

Solution: Let $T_t = \sum_{k=10^{t-1}}^{10^t-1} 2^{f(k)}$. Then clearly $S_n = \sum_{t=1}^{10} T_t$. In the sum defining $T_t$, $k$ ranges through all the whole numbers with t digits. Let $N_r$ be the number of these t-digit integers having exactly $r$ zeros for $r = 0,1,2, \ldots, t-1$. Then $T_t = \sum_{r=0}^{t-1} N_r 2^r$. We show that $N_r = {}^{t-1}C_r \times 9^{t-r}$. Indeed, since the first digit cannot be 0, the factor ${}^{t-1}C_r$ is the number of ways of choosing in which $r$ of the remaining $t - 1$ positions to place the $r$ zeros. Whichever choice is made, there remain $t - r$ positions each to be filled by one of $\{1, 2, \ldots, 9\}$, which can be done in $9^{t-r}$ different ways. Hence

$$T_t = \sum_{r=0}^{t-1} {}^{t-1}C_r\, 9^{t-r}\, 2^r = 9^t \sum_{r=0}^{t-1} {}^{t-1}C_r \left(\frac{2}{9}\right)^r$$

$$= 9^t\left(1 + \frac{2}{9}\right)^{t-1} = 9.11^{t-1};$$

and finally

$$S_n = \sum_{t=1}^{10} T_t = 9 \sum_{t=1}^{10} 11^{t-1} = 9\left(\frac{11^{10} - 1}{11 - 1}\right)$$

$$= \frac{9}{10}(11^{10} - 1).$$

Q. 668 Which is larger $8^{8^{8^{\cdot^{\cdot^{\cdot^{8}}}}}}$ (where there are altogether nine 8's in the tower), or $9^{9^{\cdot^{\cdot^{\cdot^{9}}}}}$ (where there are eight 9's)?

Solution: $8^{8^{8^{\cdot^{\cdot^{8}}}}}$ (with nine 8's) is much larger than $9^{9^{\cdot^{\cdot^{9}}}}$ (with eight 9's).

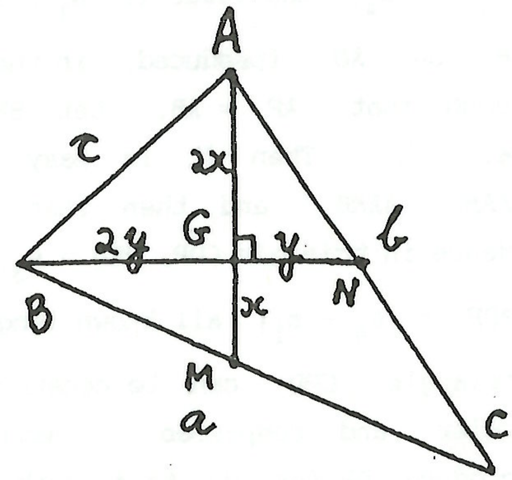
One way to prove this is to note first that if m,n are positive integers with $m > 2n$ then $8^m > 2.9^n$.

(Proof: $8^m > 8^{2n} = (\frac{64}{9})^n.9^n > 2.9^n$.)

Hence since $8^8 > 2.9$ we have immediately $8^{8^8} > 2.9^9$, and then $8^{8^{8^8}} > 2.9^{9^9}$ and so on. Eventually, (however many times we repeat the process) the value of a tower containing t8's will be over twice as large as the tower containing (t − 1)9's.

Q. 669 The medians to two sides of a triangle meet at right angles. The two sides have lengths a and b units. Find conditions on a and b for this to be possible, and express the length of the third side in terms of a and b.

Solution: Let G be the centroid, and let the distances from G to the mid points of the sides of length a,b be x,y respectively. Since G is a point of trisection of the medians, the distances from G to the vertices A,B opposite those sides are 2x, 2y respectively. Using Pythagoras theorem we obtain $(\frac{a}{2})^2 = x^2 + 4y^2$ from ΔBGM i.e.




$$a^2 = 4x^2 + 16y^2$$

Similarly

$$b^2 = 16x^2 + 4y^2 \text{ from } \Delta AGN$$

and

$$AB^2 = c^2 = 4x^2 + 4y^2 \text{ from } \Delta AGB.$$

It is clear that $c^2 < a^2$ and $c^2 < b^2$ i.e. c is the shortest side. To answer the last part first, we have $c^2 = \frac{1}{5}(a^2 + b^2)$.

Now a triangle with side lengths a,b,c can exist only if $c > |a - b|$; i.e. $c^2 > (a - b)^2$. Using $c^2 = \frac{1}{5}|a^2 + b^2|$ this simplifies to

$$2a^2 - 5ab + 2b^2 < 0$$

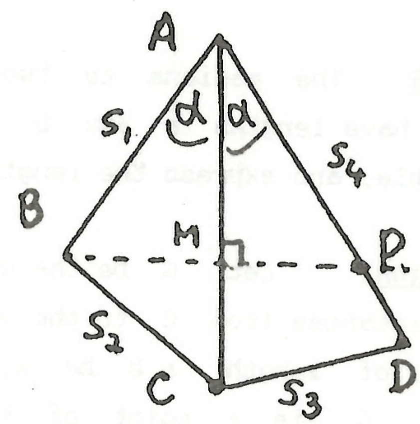
$$(a - 2b)(2a - b) < 0.$$

The factors $a - 2b$ and $2a - b$ must have opposite signs. This is the case if and only if

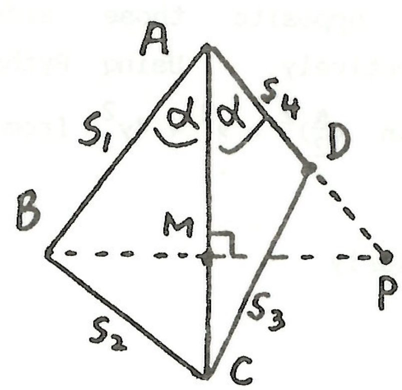
$$\frac{a}{2} < b < 2a;$$

i.e. provided the ratio of the side lengths, $\frac{a}{b}$, is between $\frac{1}{2}$ and 2.

Q. 670 Show how to construct a quadrilateral ABCD given that the diagonal AC bisects the angle A, and given also the lengths of the four sides.



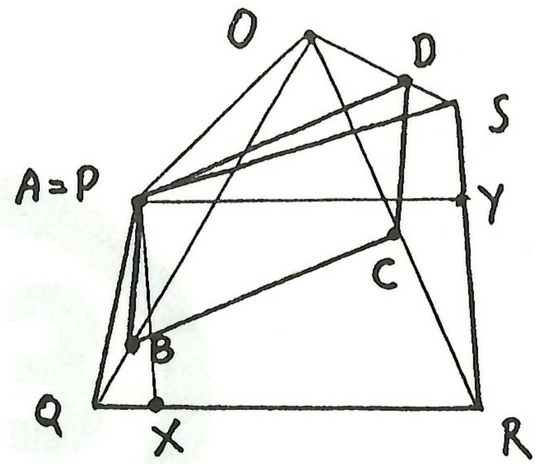

Solution: Let the figure show the desired quadrilateral, in which $s_1, s_2, s_3, s_4$ are given lengths. (The upper diagram applies if $s_1 < s_4$, the lower if $s_1 > s_4$.) Construct P on AD (produced, in the lower diagram) such that AP = AB. Let BP intersect AC at M. Then it is easy to prove that $\Delta AMP \equiv \Delta AMB$, and then that $\Delta CMP \equiv \Delta CMB$. Hence in triangle CDP, $*CD = s_3$, $*CP = s_2$, and $*DP = |s_4 - s_1|$ all known lengths. Thus the triangle CPD can be constructed first with ruler and compasses. Having done this, produce DP (or PD) to A such that $*DA = s_4$. Join AC and construct B as the mirror image of P in AC. (Clearly the construction will be possible from the 4 given lengths if it is possible to construct a triangle with sides of lengths $s_2$, $s_3$ and $|s_1 \; s_4|$, or if $s_1 = s_4$ and $s_2 = s_3$.)



Q. 671 Four rays in 3-D space are drawn from 0 to the vertices of a convex quadrilateral. Show that there exists a parallelogram ABCD having one vertex on each ray.

Solution: Let the given quadrilateral be PQRS, with the vertices labelled such that there exist points X,Y on the sides QR, RS (not produced) with $PX \parallel SR$ and $PY \parallel QR$. [If PQRS already has one pair of sides parallel, either X or Y will coincide with a vertex. The following argument simplifies in this case, but we shall not give the appropriate re-wording. We also leave it to you to observe that every convex quadrilateral without parallel sides can have its vertices labelled as stipulated.]

Take A co-incident with P. Let B be the point on OQ (see figure) such that $\frac{QB}{BO} = \frac{QX}{XR}$, and D the point on OS such that $\frac{SD}{DO} = \frac{SY}{YR}$. By our construction $BX \parallel OR$ as well as $AX \parallel RS$. Hence plane ABX $\parallel$ plane ORS. Similarly, plane ADY $\parallel$ plane OQR. From this, since AD $\parallel$ plane OQR, there is a line through B $\parallel$ AD which lies entirely in plane OQR. Let this line intersect OR at C. Join CD.




Now since $BC \parallel AD$, ABCD is a plane figure. Since AB and CD lie in the parallel planes ABX and ORS, they can never meet, and must be parallel lines. Thus ABCD is a parallelogram.

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