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POLYGONS, POLYHEDRA & PYTHAGORAS

by

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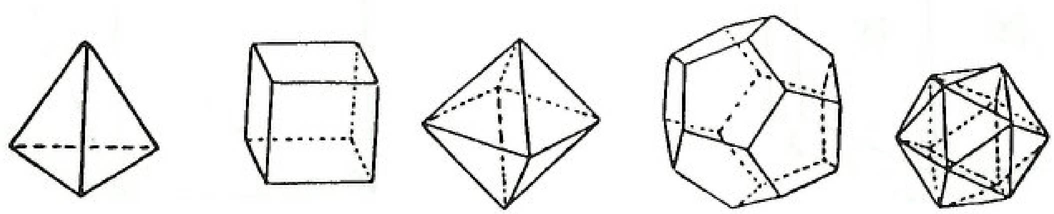
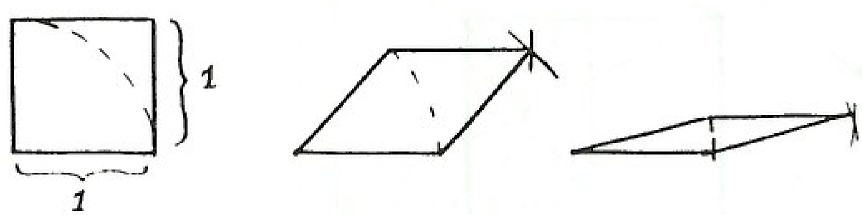
Architecture and geometry are old friends. Often, architects [7, 8] borrow interesting shapes from geometers, but traffic in the other direction is less frequent. This is the story of a geometric discovery which was accidentally uncovered in an investigation of space frames.

All five Platonic solids [3, 5] make good modules for space frames, but the strength and rigidity depends on the shape and on how the frame is constructed.



To illustrate the problem, consider a square made of four equal length rods joined by flexible connectors at the corners. This can be distorted to take the shape of a rhombus or even a skew quadrilateral.



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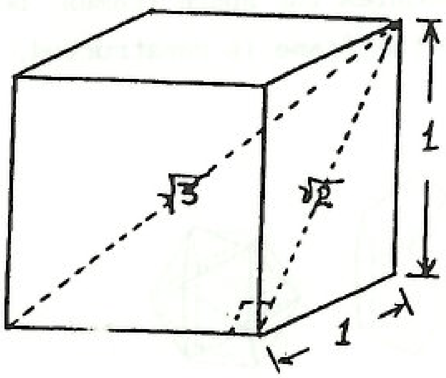
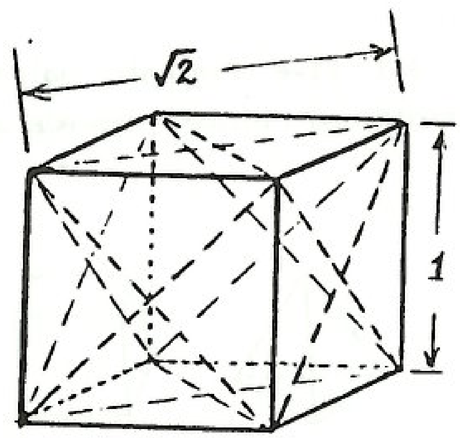
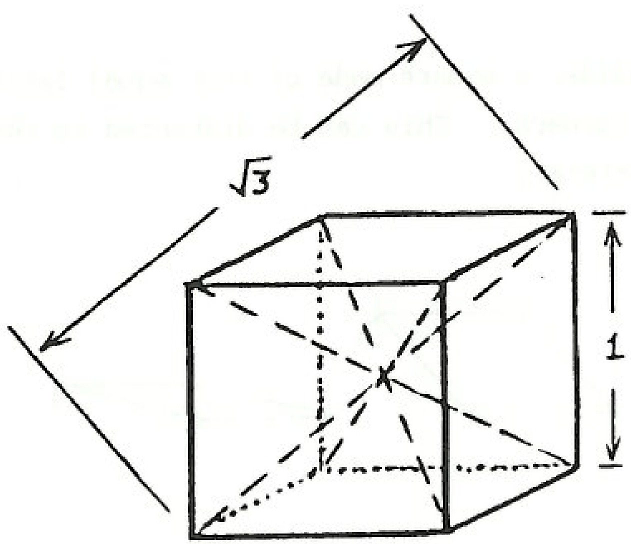
But it is easy to prevent this by stringing cables of length $\sqrt{2}$ between diagonally opposite vertices. This prevents the opposite vertices from getting more than $\sqrt{2}$ apart (though they can get closer together). Since neither pair can extend, both are forced to remain $\sqrt{2}$ apart. The result is a rigid square frame. A cube made of rigid rods of length 1 joined by flexible connectors is also not rigid and can easily collapse flat on its face. If each square face is braced by two diagonal cables of length $\sqrt{2}$, the cube becomes rigid. What if internal diagonal cables of length $\sqrt{3}$ are substituted for the face cables? Does it make the cube rigid? Make a model and see whether it falls flat on its face.





EXTERNAL DIAGONALS



INTERNAL DIAGONALS

There are many ways to make a cube rigid and it is useful to have an inventory of parts to compare various frameworks.

Component	length	number of components			
	ℓ	f	$f\ell$	ℓ^2	$f\ell^2$
edge rod	1	12	12.00	1	12
face cable	$\sqrt{2}$	12	16.97	2	24
inner cable	$\sqrt{3}$	4	6.93	3	12
sums		28	35.90		48

$$\text{mean length of components} = \frac{35.90}{28} = 1.28$$

$$\text{mean square length of components} = \frac{48}{28} = 1.71$$

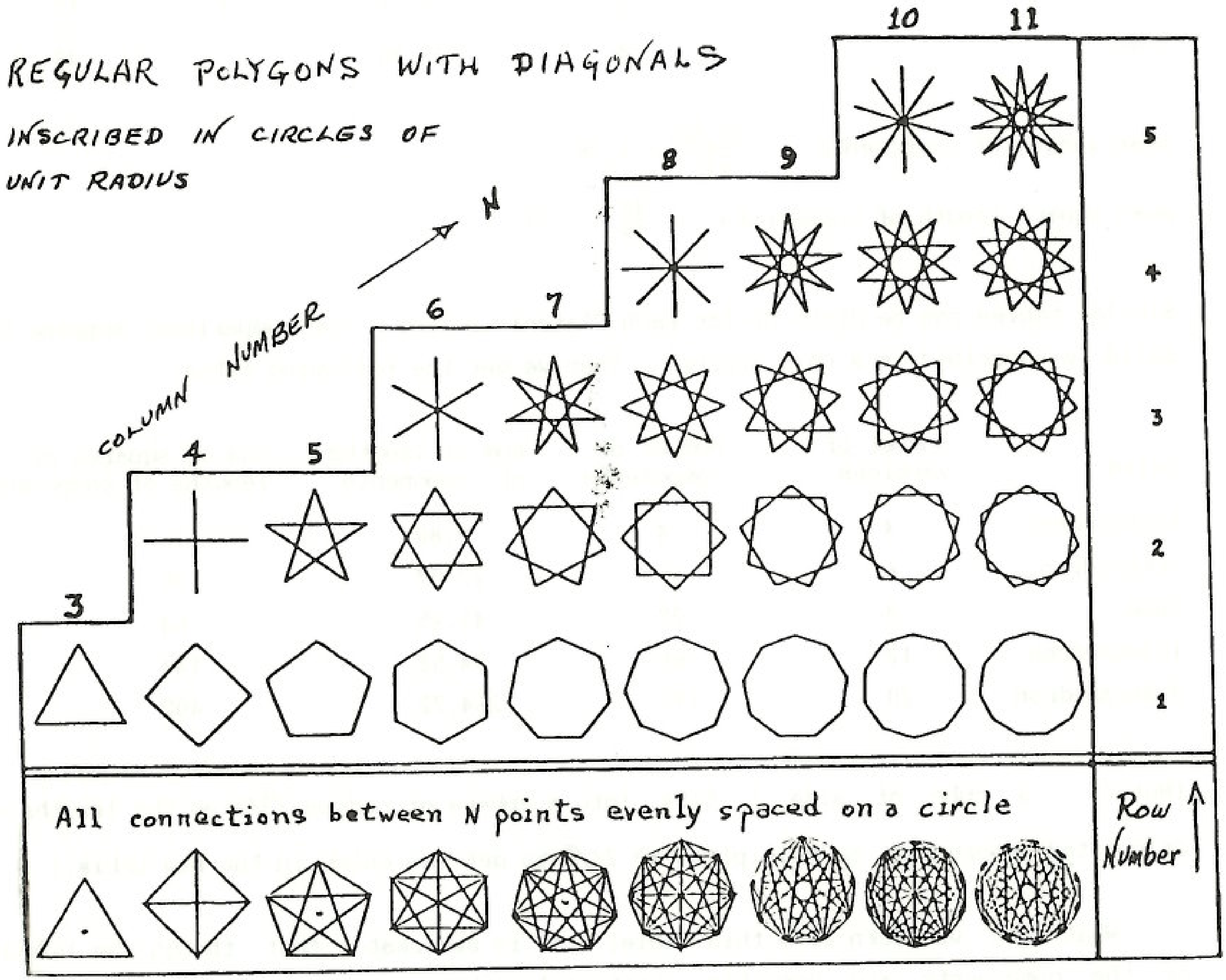
Similar tables can be drawn up for each Platonic solid. For comparison, suppose the solid is inscribed in a unit sphere. Then we get the following table.

Solid	number of vertices	number of components	sum of lengths of components	sum of squares of lengths of components
tetrahedron	4	4	9.80	16
octahedron	6	15	22.97	36
cube	8	28	41.45	64
icosahedron	12	66	94.58	144
dodecahedron	20	190	264.72	400

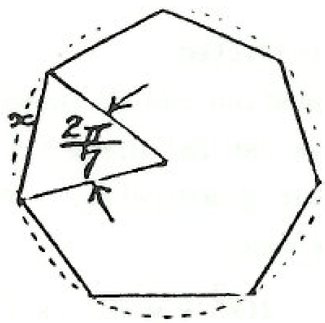
(Note: a cube of side 1 fits into a sphere of radius $\sqrt{3}/2$ so the lengths we calculated before must be multiplied by $2/\sqrt{3}$ to get the entry in the new table.)

What can we learn from this table? It is remarkable that, though the lengths of the components are horrible irrational numbers, the sum of the squares of the lengths of the components is always an integer. What is more, it is the square of the number of vertices. It seems reasonable to make the following guess. Suppose we have a polyhedron with N vertices all of which lie on a sphere of radius r. Then the sum of the squares of all the distances between pairs of vertices is N^2R^2. (This is a sum of $\binom{N}{2} = \frac{1}{2}N(N - 1)$ terms.) Our guess is true for all Platonic solids as the table above shows. Is it true for others?

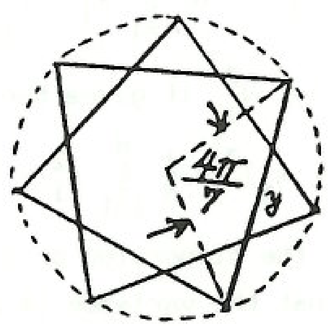
The calculations are quite laborious for complicated polyhedra. It makes sense to look for more evidence in a simpler case and so we try some calculations for polygons. (In the same way a man who lost his wallet at night would first look for it near a streetlight.) The next figure shows various polygons and the lengths between pairs of vertices which we have to sum [6].



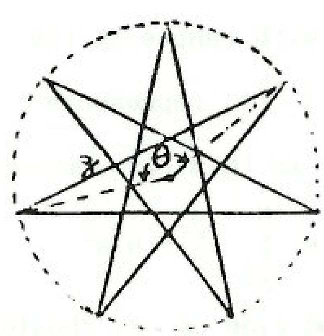
Here are the calculations for the regular heptagon. We take 7 points equally spaced around a circle of radius 1, so $N = 7$ and $R = 1$. There are three sorts of chords between pairs of vertices.



A chord joining two neighbouring vertices forms an angle of $2\pi/7$ radians (or 360/7 degrees) at the centre of the circle and so its length is $x = 2 \sin \pi/7$. There are 7 chords like this.



Next there are 7 chords obtained by joining each point to its second nearest neighbour. Each of these has length $y = 2 \sin 2\pi/7$.



$\theta = 6\pi/7$

Finally, there are 7 chords obtained by joining each point to its third nearest neighbour. Each of these has length $z = 2\sin 3\pi/7$. The sum of the squares of all these lengths is

$$7(x^2 + y^2 + z^2) = 7(4\sin^2\frac{\pi}{7} + 4\sin^2\frac{2\pi}{7} + 4\sin^2\frac{3\pi}{7}) = 49,$$

that is exactly $7^2 1^2 = N^2R^2$, as predicted.

Why is it so? The explanation can be found in a calculation in elementary statistics. The point is that we can calculate the distance between two points by finding the difference between their coordinates. There are some useful identities involving sum of squares of differences:

$$(x - y)^2 = 2(x^2 + y^2) - (x + y)^2,$$

$$(x - y)^2 + (x - z)^2 + (y - z)^2 = 3(x^2 + y^2 + z^2) - (x + y + z)^2,$$

$$(x - y)^2 + (x - z)^2 + (x - w)^2 + (y - z)^2 + (y - w)^2 + (z - w)^2$$
$$= 4(x^2 + y^2 + z^2 + w^2) - (x + y + z + w)^2$$

and, in general,

$$\sum_{1\leq i<j\leq N}^{N} (x_i - x_j)^2 = N\sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2. \qquad (*)$$

When the last equation is divided by N^2 it gives the well-known equation

$$\frac{1}{N^2}\sum_{1\leq i<j\leq N} (x_i - x_j)^2 = \frac{1}{N}\sum_{i=1}^{N} x_i^2 - (\frac{1}{N}\sum_{i=1}^{N} x_i)^2.$$

The left hand side is a measure of the dispersion or scatter of the x's, called the variance. The formula shows that the variance is equal to the mean square of the x's minus the square of the mean, and this gives a much more economical way to calculate it. Now back to our geometrical problem. The same algebra works in 2 or 3 dimensions, when we interpret the x's as coordinates of the vertices of a polygon or a polyhedron. The left-hand side of (*) is the sum of the squares of the distances between all the pairs of vertices. Suppose all the vertices lie on a circle (or sphere) of radius R with centre at the origin. Each x_i^2 in the first term on the right-hand side of (*) represents the square of the distance from a vertex to the origin, namely R^2, so this term becomes

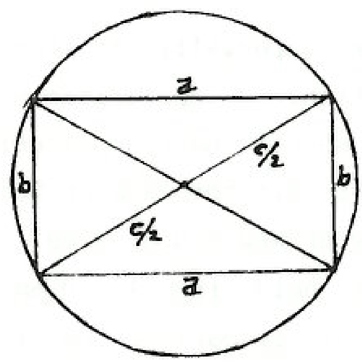
$$N\sum_{i=1}^{N} x_i^2 = N^2R^2.$$

The second term on the right-hand side of (*) should not be there! Alternatively, our guess is not quite correct. We can interpret

$$Z = \frac{1}{N}\sum_{i=1}^{N} x_i$$

as the average location of the vertices, that is the centre of gravity [4] of the polygon (or polyhedron). For a regular polygon inscribed in a circle, the centroid is at the centre (origin), so $Z = 0$ and the sum of the squares of the distances between all pairs of vertices is indeed N^2R^2. In general, we have the theorem: the sum of the squares of the distances between all pairs of vertices of a $\{\begin{matrix}\text{polygon}\\ \text{polyhedron}\end{matrix}\}$ inscribed in a $\{\begin{matrix}\text{circle}\\ \text{sphere}\end{matrix}\}$ of radius R is $N^2R^2 - N^2Z^2$, where Z is the distance between the centre of the circle and the centre of gravity of the $\{\begin{matrix}\text{polygon}\\ \text{polyhedron}\end{matrix}\}$.

Here is a final special case. We have $N = 4$ points which are ends of two diameters of a circle of radius $c/2$. These points are connected by six line segments of length a, b and c as shown.



Since the centroid of the 4 points is at the centre, $z = 0$ and by the theorem the sum of the squares of distances between all 6 pairs of points is

$N^2R^2 = 4^2\frac{c}{2}^2 = 4c^2$. Also, this sum of squares equals $2a^2 + 2b^2 + 2c^2$. Equating these two sums gives

$$a^2 + b^2 = c^2.$$

We have rediscovered Pythagoras' theorem! (The triangle with sides of length a, b and c is a right-triangle because c is a diameter of the circle.)

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The University of New South Wales 26th School Mathematics Competition Thursday 18th June 1987

Readers of Parabola are especially invited to consider entering this years School Mathematics Competition. Applications must be made through School Principals, all of whom should shortly have application forms and information. Some schools will arrange for you to sit the exam at School, while others will allow you to sit for it at the University of New South Wales.

The examination will be held from 10:00 am to 1:00 pm on Thursday 18th June 1987. You will notice that this is earlier than in past years due to the change to a four term year. As in previous years there will be a Junior Division (Year 10 or lower and under 17 on 30th June) and Senior Division (at high school and under 19). Each division has cash prizes of $100, $75, $50 and up to twenty at $20, all generously donated by IBM Australia Ltd. and knows as "IBM Prizes". The prize winners and next forty students will receive a certificate of merit.

The competition is so designed that mathematical insight and ingenuity are needed for success, rather than efficiency in tackling routine examples. Students are particularly encouraged to enter the competition if they are able to make some progress towards the solution of at least one of the problems from the 1986 competition. (The problems and their solutions are printed in Parabola in No.2 of each Volume so look in back issues for them.) Entrants are allowed to take any books and materials including electronic calculators into the examination.

So, make sure you contact your Principal in late March or early April to get an application form. Closing date is 8th May at the University and earlier at individual schools. Any enquiries concerning the competition should be directed to the University of New South Wales, Examinations Section, phone 697-3085.