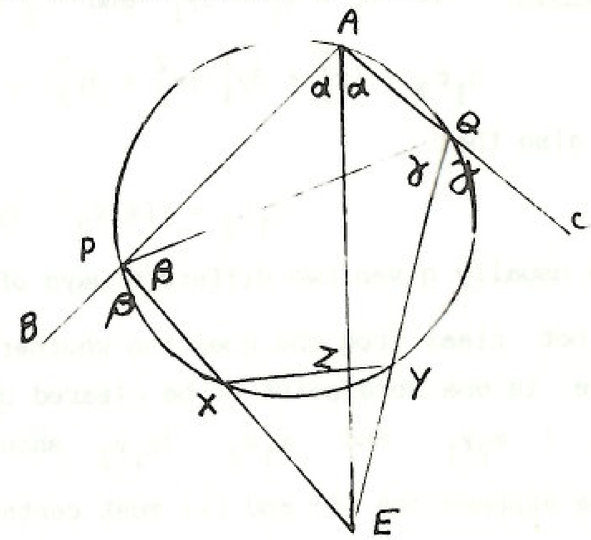
SOLUTIONS TO PROBLEMS FROM VOLUME 22, NUMBER 3

Q. 684. $B\hat{A}C$ is an obtuse angle. A circle through A cuts AB at P and AC at Q. The bisectors of angles $Q\hat{P}B$ and $P\hat{Q}C$ cut the circle at X and Y respectively. Prove that XY is perpendicular to the bisector of $B\hat{A}C$.



Solution. Let the bisectors of $B\hat{P}Q$ and $C\hat{Q}P$ meet at E. Then E is equidistant from BP and PQ, and from QP and QC. Hence E is equidistant from AB and AC, so also lies on the bisector of $B\hat{A}C$. (It is of course well known that these three bisectors are concurrent at an "exocentre" of the triangle APQ.) Let α, β, γ be angles equal to halves of $P\hat{A}C$, $B\hat{P}Q$ and $P\hat{Q}C$ respectively as in the figure. Since the angles of ΔAPQ (viz 2α, $180^\circ - 2\beta$, and $180^\circ - 2\gamma$) sum to 180°, we easily obtain $\beta + \gamma - \alpha = 90^\circ$. Since the angles of the quadrilateral AZYQ sum to 360° we obtain $A\hat{Z}Y = 360^\circ - Z\hat{A}Q - A\hat{Q}Y - Q\hat{Y}Z$. But $Z\hat{A}Q = \alpha$, $A\hat{Q}Y = 180^\circ - \gamma$ (since AQC is straight) and $Q\hat{Y}Z = 180^\circ - Q\hat{P}X$ (since XYQP is cyclic).

$$= 180^\circ - \beta$$

Hence $A\hat{Z}Y = 360^\circ - \alpha - (180 - \gamma) - (180 - \beta) = \beta + \gamma - \alpha = 90^\circ$.

Q. 685. Solve the equation

$$9^{x+1} - 5^{2x+1} = 3^{2x-1} + 5^{2x-1}.$$

Solution. $3^{2x+2} - 3^{2x-1} = 5^{2x+1} + 5^{2x-1}$

$\Leftrightarrow 3^{2x-1}(27 - 1) = 5^{2x-1}(5^2 + 1)$

$\Leftrightarrow (\frac{3}{5})^{2x-1} = 1$

$\Leftrightarrow 2x - 1 = 0$. Therefore $x = \frac{1}{2}$ is the only solution.

Q. 686. Both c_1 and c_2 are expressible in the form $c = x^2 + 3y^2$ where x and y are whole numbers. Prove that their product c_1c_2 is also expressible in that form.

[e.g. $c_1 = 13 = 1^2 + 3.2^2$: $c_2 = 12 = 3^2 + 3.1^2$ and $156 = 12.13 = 9^2 + 3 \times 5^2$.]

Solution. If $c_1 = x_1^2 + 3y_1^2$ and $c_2 = x_2^2 + 3y_2^2$ one can check immediately that

$$c_1c_2 = (x_1^2 + 3y_1^2)(x_2^2 + 3y_2^2) = (x_1x_2 + 3y_1y_2)^2 + 3|(x_1y_2 - x_2y_1)|^2 \qquad (*)$$

and also that

$$c_1c_2 = |(x_1x_2 - 3y_1y_2)|^2 + 3(x_1y_2 + x_2y_1)^2 \qquad (+)$$

This usually gives two different ways of expressing c_1c_2 in the required form. It is not clear from the question whether 0 is allowed as a whole number, and if not there is one more point to be cleared up. We shall show that it is impossible that $x_1y_2 - x_2y_1$ and $x_1x_2 - 3y_1y_2$ should both be zero, and then at least one of the above expressions (*) and (+) must certainly be acceptable. Suppose

$$x_1y_2 - x_2y_1 = 0 \text{ and } x_1x_2 - 3y_1y_2 = 0.$$

Then

$$x_1y_2\, x_1x_2 = x_2y_1\, 3y_1y_2$$

$$x_1^2 = 3y_1^2 \text{ and } \frac{x_1}{y_1} = \sqrt{3}.$$

But since $\sqrt{3}$ is known to be irrational, this is impossible.

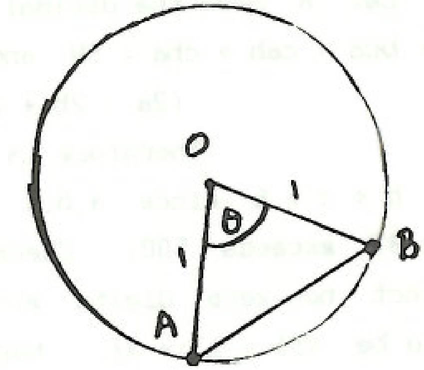
Q. 687. Find all integers x,y,z such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x + y + z}.$$

Solution. $\frac{1}{x} + \frac{1}{y} = \frac{1}{x + y + z} - \frac{1}{z}$ iff $\frac{x + y}{x \times y} = \frac{-(x + y)}{z(x + y + z)}$

This is true if $x + y = 0$, any non zero x,y,z, or if $z^2 + zy + zx + xy = 0$ i.e. $(z + x)(z + y) = 0$ whence $x + z = 0$ or $y + z = 0$. Thus all integer solutions are given by $(x,y,z) = (\lambda, -\lambda, \mu)$ or $(x,y,z) = (\lambda, \mu, -\lambda)$ or $(x,y,z) = (\mu, \lambda, -\lambda)$ where λ,μ are any non zero integers.

Q. 688. A regular polygon with n vertices is inscribed in a circle of radius 1. Let L be the set of all <u>mutually distinct</u> lengths of all line segments joining the vertices of the polygon. What is the sum of the squares of the elements of L?



Solution. In the figure, by the cosine rule

$AB^2 = 1^2+1^2 - 2.1.1. \cos\theta = 2 - 2\cos\theta$

For a regular n-gon inscribed in a unit circle, the joins of vertices subtend angles at the centre equal to $k\frac{2\pi}{n}$ for

$k = 1, 2, \ldots, \frac{n}{2}$ (n even) or

$k = 1, 2, \ldots, \frac{n-1}{2}$ (n odd). Hence the sum of the squares of the elements of L is

$$S = (2 - 2\cos\theta) + (2 - 2\cos 2\theta) + \ldots + (2 - 2\cos\tfrac{n}{2}\theta) \quad \text{for } n \text{ even}$$

or

$$S = (2 - 2\cos\theta) + (2 - 2\cos 2\theta) + \ldots + (2 - 2\cos\tfrac{(n-1)}{2}\theta) \quad \text{for } n \text{ odd},$$

where $\theta = \frac{2\pi}{n}$.

$$S = n - (2\cos\theta + 2\cos 2\theta + \ldots + 2\cos\tfrac{n}{2}\theta) \quad (n \text{ even})$$

or

$$S = (n-1) - (2\cos\theta + \ldots + 2\cos\tfrac{n-1}{2}\,\theta) \quad (n \text{ odd}).$$

Setting $T = 2\cos\theta + 2\cos 2\theta + \ldots + 2\cos m\theta$.

$$T\sin\frac{\theta}{2} = (\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}) + (\sin\frac{5\theta}{2} - \sin\frac{3\theta}{2}) + \ldots + (\sin(m+\tfrac{1}{2})\theta - \sin(m-\tfrac{1}{2})\theta)$$

$$= \sin(m+\tfrac{1}{2})\theta - \sin\frac{\theta}{2}.$$

Putting $\theta = \frac{2\pi}{n}$, $m = \frac{n}{2}$ we obtain $T = \dfrac{\sin(\pi + \frac{\pi}{n}) - \sin\frac{\pi}{n}}{\sin\frac{\pi}{x}} = -2$ and

$S = n - (-2) = n + 2$ when n is even.

Putting $m = \frac{n-1}{2}$ we obtain $T = \dfrac{\sin\pi - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}} = -1$ and $S = (n-1) - (-1) = n$

when n is odd.

Q. 689. N is a number less than 500 with three distinct digits, none of them 0. Five different numbers can be obtained by changing the order of the digits of N. The arithmetic mean of these five numbers is equal to N. Find N.

Solution. Let N have the decimal representation abc. Then acb + bac + bca + cab + cba = 5N and (adding abc = N to the equation)

$$(2a + 2b + 2c)(100 + 10 + 1) = 6N.$$

$$\text{Therefore } (a + b + c) \times 37 = N = abc.$$

Now $a + b + c > 6$ since a,b,c are distinct non-zero digits and $a + b + c < 13$ since 14×37 exceeds 500. Checking these eight multiplies of 37, the only one with distinct non-zero digits such that the other factor is the sum of the digits turns out to be $481 = 13 \times 37$. Hence $N = 481$.

Q. 690. i) Is there a natural number n such that $n^2 + n + 1$ is exactly divisible by 1985, or by 1986?

ii) Is there a natural number n such that $2(n^2 + n + 1)$ is divisible by 1986?

Solution. i) Whether n is even or odd, $n^2 + n + 1$ is odd and cannot be a multiple of 1986. If the last digit of n is 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 the last digit of $n^2 + n + 1$ is 1, 3, 7, 3, 1, 6, 3, 7, 3, or 1 respectively. Hence $n^2 + n + 1$ cannot be a multiple of 1985.

ii) Since 1986 is itself equal to $2 \times (31^2 + 31 + 1)$ the answer is clearly yes.

Q. 691. Add 3 digits to the end of 523 so that the resulting 6 digit number is divisible by 7,8, and 9.

Solution. Since on dividing 523000 by each of 7, 8, and 9 in turn the remainders are 2, 0, and 1 respectively, we wish to add on a three digit number abc which leaves remainders 5, 0 and 8 respectively when divided by 7, 8, and 9. The smallest multiple of 8 which leaves a remainder of 5 when divided by 7 is $5 \times 8 = 40$. If one adds any multiple of 56 ($= 7 \times 8$) to this, these two remainders are unaltered. Since 40 leaves remainder 4 on division by 9 and 56 leaves the remainder 2 on division by 9,

$40 + k \times 56$ exceeds a multiple of 9 by $4 + 2k$.

This is equal to 8 when $k = 2$. Therefore $152 = 40 + 2 \times 56$ has all the right remainders and hence 523152 is a solution. Since $7 \times 8 \times 9 = 504$, $152 + 504 = 656$ has the same remainders as 152, and a second possible answer is 523656.

Q. 692. Seven colours of paint are available. In how many ways can the 6 faces of a rectangular block be painted if faces meeting at an edge must be given different colours.

Solution. We assume either that the block is fixed in space (or alternatively has faces which remain identifiable after painting). Afterwards we consider what difference occurs in the answer if the block is moveable and opposite faces are indistinguishable if both are painted with the same colour.

If all three pairs of opposite faces are similarly coloured, there are $7 \times 6 \times 5$ ways of painting the block. (For each of 7 ways of choosing a colour for the base and top, there are 6 ways of choosing one of the remaining colours for the East and West faces, and then 5 ways of choosing one of the still unused colours for the North and South faces.)

If two pairs of opposite faces are similarly coloured, there are $3 \times 7 \times 6 \times 5 \times 4$ ways of painting the block. (If, say the base and top have the same colour, and also the East and West faces, but not the North and South faces, there are $7 \times 6 \times 5 \times 4$ ways of assigning colours to these regions. A similar calculation applies if the similarly coloured opposite faces are base and top together with North and south faces, or east and West faces together with North and South faces; hence the factor of 3.)

If one pair of opposite faces are similarly coloured, there are $3 \times 7 \times 6 \times 5 \times 4 \times 3$ ways of painting the block.

If no pair of opposite faces are similarly coloured, the faces can be assigned six different colours in $7 \times 6 \times 5 \times 4 \times 3 \times 2$ ways.

Hence the total number of colourings is the sum of these four results

$$= 7 \times 6 \times 5(1 + 12 + 36 + 24) = \underline{15,330}.$$

Now if the block is moveable (but we assume that the side lengths a,b,c are all different), a smaller answer may be given since some different paintings yield apparently identical results. For example, if two pairs of opposite faces are similarly coloured and it is decided to paint the remaining pair (say the North and South faces) red and yellow it makes no difference which is painted red and which yellow, since for example it could be rotated 180° about a vertical axis.

Thus the second term in the above calculation should be divided by 2 if the number of indistinguishable paintings is required. Similarly you may check that the third and fourth terms should each be divided by four because if either 1 pair or 0 pair of opposite faces is similarly coloured, there are 4 different paintings with the chosen colours yielding identical blocks.

Thus the answer in this case is 7.6.5 (1 + 6 + 9+ 6) = 4620.

Q. 693. N is a power of 2, and $(a_1, a_2, \ldots, a_N)$ is a sequence of N numbers each equal to either $+1$ or -1. Another sequence (b_i) of the same length is obtained by the following operation:-

$$b_i = a_i a_{i+1} \quad i = i, \ldots, N-1 \quad ; \quad b_N = a_N a_1.$$

This operation is performed repeatedly. Prove that after enough repetitions of the operation all terms of the resulting sequence are equal to $+1$.

Solution. Let $k_1, k_2, \ldots, k_N$ be the sequence obtained after the process has been performed m times. For example if

$$m = 1 \quad k_1 = a_1a_2,\ k_2 = a_2a_3,\ \ldots,\ k_t = a_ta_{t+1},\ \ldots$$

$$m = 2 \quad k_1 = a_1a_2^2a_3,\ k_2 = a_2a_3^2a_4,\ \ldots,\ k_t = a_ta_{t+1}^2a_{t+2},\ \ldots$$

$$m = 3 \quad k_1 = a_1a_2^3a_3^3a_4,\ k_2 = a_2a_3^3a_4^3a_5,\ \ldots,\ k_t = a_ta_{t+1}^3a_{t+2}^3a_{t+3}.$$

(In these expressions, a_{N+r} is used as an alternative name for a_r for $r = 1,2, \ldots$)

Now one recognizes the list of exponents in each of the terms corresponding to some m as the numbers in the mth row of Pascal's triangle (which are known to be equal to the binomial coefficients ${}^mC_0, {}^mC_1, \ldots, {}^mC_m$.) e.g. If $m = 3$, the exponents in the expression for k_t are 1, 3, 3, 1, which are the binomial coefficients ${}^3C_0$, ${}^3C_1$, ${}^3C_2$, ${}^3C_3$. [It is evident that this pattern persists for any m, for suppose after m operations

$$k_t = a_ta_{t+1}^{{}^mC_1}a_{t+2}^{{}^mC_2} \cdots a_{t+m}^{{}^mC_m},\quad k_{t+1} = a_{t+1}a_{t+1}^{{}^mC_1} \cdots a_{t+m+1}^{{}^mC_m}.$$

Then the tth term in the sequence obtained after once more performing the given operation is

$$a_ta_{t+1}^{({}^mC_2+1)}a_{t+2}^{({}^mC_2+{}^mC_1)} \cdots a_{t+r}^{({}^mC_2+{}^mC_{r-1})} \cdots a_{t+m+1}.$$

Since $^{m}C_{r} + {}^{m}C_{r-1} = {}^{m+1}C_{r}$ (the "Pascal Triangle Property" of the binomial coefficients), this term is

$$a_t a_{t+1}^{\,{}^{m+1}C_1} a_{t+2}^{\,{}^{m+1}C_2} \dots a_{t+r}^{\,{}^{m+1}C_r} \dots a_{t+m+1}.]$$

Now any row of Pascal's triangle begins and ends with 1, but for some rows all of the other entries are even numbers. e.g. the fourth row is 1 4 6 4 1. You can easily check that the same is true for the second row, and the eighth row. (See figure.) In fact, if N is a power of 2, all members in the Nth row of Pascals triangle are even except the 1's at the ends. One way to prove this is to use the formula

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

....

$$^{N}C_{r} = \frac{N(N-1)(N-2)\dots(N-\overline{r-1})}{1 \quad 2 \qquad\qquad (r-1)r} \qquad 1 < r < N-1.$$

When N is a power of 2, it is not difficult to prove that for each integer $s(< N)$, s and $N - s$ contain the factor 2 to the same power, and hence that since N contains 2 to a larger power than r, $^{N}C_{r}$ must be even.

The final step can now be taken. Perform the given operation N times. A typical element in the resulting sequence is then

$$k_t = a_t a_{t+1}^{\,{}^{N}C_1} a_{t+2}^{\,{}^{N}C_2} \dots a_{t+N}^{1}.$$

Since each a_i is ± 1 and all exponents $^{N}C_{r}$ are even this gives $k_t = a_t a_{t+N}$.

But $a_{t+N} = a_t$, so $k_t = a_t^2 = +1$. QED.

Q. 694. A "Latin square" of order n is a square array of n lines and columns each of which consists of an arrangement of the same n symbols. The figure is an example of a Latin square of order 3 (using the symbols 1, 2, 3). It is a "symmetric" Latin square i.e. symmetric about the "main" diagonal from the 1st element of

1	2	3
2	3	1
3	1	2

the top row to the last element in the bottom row. This means that for each i, j the element in the ith row and jth column is the same as that in the jth row and ith column.

Prove that on the main diagonal of every symmetric Latin square of odd order each of the n symbols occurs once.

__Solution.__ Suppose some symbol k does not occur on the main diagonal. Because of the symmetry k occurs the same number of times above the main diagonal as below, so it occurs an even number of times altogether. But this is impossible since it occurs exactly once in each of the n rows, and n is odd. Thus the main diagonal must contain each of the n symbols.

__Q. 695.__ Suppose $p(x)$ denotes a polynomial $a_0 + a_1x + \ldots + a_nx^n$ which is such that the equation $p(x) = x$ has no real roots. Show that the equation $p(p(x)) = x$ also has no real roots.

__Solution.__ If the polynomial $p(x) - x$ never vanishes its values must either be all positive or all negative. (If say $p(x_1) - x_1 > 0$ and $p(x_2) - x_2 < 0$ then there must be some point x_3 between x_1 and x_2 at which the graph of $p(x) - x$ crosses the x axis, i.e. $p(x_3) - x_3 = 0$.) Suppose $p(x) - x > 0$ for all x. Then, replacing x by $p(x)$ everywhere, $p(p(x)) - p(x) > 0$ for all x. Adding, we obtain $p(p(x) - x > 0$ for all x. An identical argument applies if $p(x) - x < 0$ for all x.

J.J.F. Hawley (Cranbrook) submitted correct solutions to Q672, Q673, a partly correct solution to Q675, both arguments for Q683 (i) and a partial solution for (ii).

D. Jackson (Sydney Grammar) submitted correct solutions to Q672, Q673, Q675, Q678, Q683, Q683, Q687, Q689.

L.A. Koe (James Ruse) gave correct arguments for both parts of Q683.

F. Antonvccio (St. Gregory's College) submitted correct solutions for Q684, Q685, Q687, Q688, Q689, Q690, Q691, Q694, Q695, and an almost complete solution of Q693 (with the omission acknowledged).

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