SOLITONS - THE WAVES OF THE FUTURE?

by
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One of the more fascinating and unexpected discoveries of modern mathematics is the soliton. A single soliton is a solitary wave, that is an isolated disturbance of permanent form which propagates with constant speed. Such waves can be observed on the surface of water, or as cloud lines in the atmosphere, or in other physical systems. The existence of solitary waves has been known for over a hundred years. However, we had to wait for the advent of the computer before the fascinating properties of the soliton were unravelled. Before the discovery of the soliton the conventional wisdom concerning waves was that waves could be divided into two categories, linear and nonlinear. Linear waves were of small amplitude (more precisely, infinitesimal amplitude), interacted by resonant mechanisms, exchanging energy until "thermalization" occurred; that is loosely speaking, after a long time any one kind of wave is as likely to be present as any other kind, while detailed knowledge of the initial state was lost. The discovery of the soliton has changed this. Solitons are nonlinear waves, which remain coherent after interaction. Indeed they interact according to well-defined nonlinear superposition rules, and have a variety of fascinating properties. Further, far from being rare phenomena of curiosity value only, they are rather common. Under certain quite general conditions, a large class of initial states will always produce solitons.

The first reported sighting of a solitary wave was made by John Scott Russell in 1834:

> "I believe I shall best introduce this phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion: it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind rolled forward with great velocity,

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assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phaenomenon ...

John Scott Russell (1845)

Russell was an eminent Victorial scientist who like other outstanding scientists of that era, was capable of combining accurate observations with imaginative experiments and simple theories, together producing considerable insight. He subsequently undertook some laboratory experiments and established that all solitary waves were waves of elevation and that the wave speed increases with wave amplitude. Thus solitary waves are nonlinear waves, since linear waves are usually oscillatory (i.e. the wave displacement can take either sign) and have wave speeds which are independent of amplitude. Russell's observations caused some controversy as the prevailing theory at that time (due to Airy) was that nonlinear waves of elevation will steepen and eventually break. Such was the pace of science in the Victorian era that it was not until the 1870's that the controversy was resolved in Russell's favour by a French mathematicician, Boussinesq, and by Rayleigh in England. They showed that the tendency for nonlinear waves to steepen (i.e. the larger amplitudes of the wave want to go faster) could be exactly belanced by the tendency for waves to disperse (i.e. the waves of shorter wavelengths want to go slower). In fact they showed that the equations of motion for water waves had an approximate solution, given by

$$\eta = a \operatorname{sech}^2 (p(x - ct))$$

where η is the free-surface displacement above the undisturbed level h (see Figure 1). x is the horizontal co-ordinate in the direction of wave propagation, and t is the time. Here $\operatorname{sech} x = (\cosh x)^{-1}$ (recall that $\cosh x = 1/2(e^x + e^{-x})$) and has the characteristic shape shown in Figure 1. Note that $\operatorname{sech} x > 0$ for all x, has a maximuim value of 1 at $x = 0$ and $\to 0$ as $|x| \to \infty$. Also note that η, given above, is a function of x and t through the single combination $x - ct$, and so describes a wave propagating to the right with speed c. Now follow the two key properties of the solitary wave.

$$\frac{c}{c_0} = 1 + \frac{a}{2h}, \quad \frac{a}{h} = \frac{4\,h^2\,p^2}{3}.$$

Here $c_0 = \sqrt{gh}$ (g is the acceleration due to gravity) and is the speed with which a wave of infinitesimal amplitude and infinite wavelength would travel. The first of these expressions shows that the speed c of a solitary wave is greater than c_0, and increases directly in proportion to the wave amplitude a. The scond expression shows that the solitary wave has infinite width, being defined for $-\infty < x < \infty$, but we can define an effective width to be 1/p).

Some two decades later in 1895, two Dutch mathematicians, Korteweg and de Vries, showed that the "sech^2"-wave, described in the previous paragraph, is a solution of the following equation

$$\frac{\partial\eta}{\partial t} + c_0\frac{\partial h}{\partial x} + \frac{3c_0}{2h}\eta\frac{\partial\eta}{\partial x} + \frac{1}{6}c_0 h^2\frac{\partial^3\eta}{\partial x^3} = 0.$$

Now η is a function of two independent variables x, t so that $\eta = \eta(x,t)$. The notation $\partial\eta/\partial t$ denotes the derivative of η with respect to t while x is kept fixed, and $\partial\eta/\partial x$ denotes the derivative of η with respect to x while t is kept fixed. This is a partial differential equation, now known as the Korteweg-de Vries equation, or KdV for short. It is a simple mater to verify that the "sech^2" -wave of the previous paragraph is indeed a solution of the KdV equation. (Note that the derivative of sech x is - sech x tanh x where tanh x = sinh x/cosh x and sinh x = $1/2(e^x - e^{-x})$). For the next seventy years that was the end of the story. The solitary wave was regarded as a curiosity, mentioned briefly in some text books and not at all in others.

All of this changed in 1965 when two mathematicians in the U.S.A., Kruskal and Zabusky, decided to have another look at the KdV equation. Being interested in the proccess by which nonlinear waves interact and exchange energy, they integrated the KdV equation numerically and found instead of "thermalization", almost total coherence. To describe their results the KdV equation is first transformed by putting $u(x,t) = 3\eta(x', t')/2h$ where $x' = h(x + 6t)$, $t' = 6\,h\,t/c_0$. The result is the canonical KdV equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 .$$

The solitary wave solution of this equation is

$$u = 2k^2 \operatorname{sech}^2 (k(x - 4k^2 t)).$$

and depends on the single parameter k: it has amplitude $2k^2$ and speed $4k^2$. Kruskal and Zabusky found numerically that an essentially arbitrary initial disturbance (i.e. $u(x,0)$) evolved into a finite number of solitary waves ordered according to their amplitudes (recall that the larger waves travel faster). Further they found that these solitary waves interacted "elastically". As a larger wave overtook a small wave, there was a nonlinear interaction, from which both waves emerged unchanged in form (i.e. with the same amplitudes and speeds). The only remnant of the interaction is a phase shift with the larger wave typically being shifted forward. Figures 2 and 3 show some computer-generated solutions of the KdV equation which exhibit these properties. Because of these particle-like propoerties of solitary waves Kruskal and Zabusky coined the word <u>soliton</u> to describe them.

Now that the computer had uncovered the phenomenon, theoreticians got to work and soon unravelled some remarkable properties of the KdV equation. There is not space to describe all of these here. However, the cornerstone of the KdV theory is the following isospectral property. Consider the ordinary differential equation for $\psi(x)$.

$$\frac{d^2\psi}{dx^2} + (\lambda + u(x,t))\,\psi = 0$$

Here $u(x,t)$ is a solution of the KdV equation and λ is a "constant" (i.e. independent of x, but may depend on t). With appropriate boundary conditions as $|x| \rightarrow B$, this equation is a linear equation for ψ which contains t as a parameter. Only certain values of λ are allowed if solutions are to exist, and these constitute the spectrum. Indeed this equation for ψ is a well-known equation occuring in a variety of physical contexts. For instance it occurs in quantum mechanics where it is known as the Schrodinger equation; ψ describes the state of an atom, u is a potential well and λ is related to the energy levels. Here, since $u(x,t)$ evolves in time, always satisfying the KdV equation, the "constant" λ is also expected to evolve in time. However, in 1967 a group of mathematicians from the U.S.A., Gardner, Greene, Kruskal and Miura, made the astonishing discovery that λ is a constant (i.e. does <u>not</u> depend on the time). Even though u(x,t) evolves with time t, λ remains constant, and is determined once and for all by the initial state, u(x,0). Further, they showed that this initial state determines N values of λ, $-k_1^2$, ----, $-k_N^2$, and that as $t \rightarrow \infty$, the solution of the KdV equation is given by

$$u(x,t) \rightarrow \sum_1^N 2\,k_n^2 \,\text{sech}^2\,[k_n\,(x - 4k_n^2\,t + x_n)], \text{ as } t \rightarrow \infty$$

This describes a train of N solitons, with amplitudes $2k_n^2$ and speeds $4k_n^2$, n = 1 --- N. In Figure 2 we see a case when N = 3. The remaining part of the solution as $t \rightarrow \infty$ consists of small scale oscillations, which propagate to the left, and ultimately decay. Thus the final state of the system consists of N solitons.

Several other properties follow. For instance, with a suitable choice for the initial state (i.e. u(x,0)) solutions can be constructed which are ccompletely free of oscillations. These are exact solutions of the KdV equation, are known as the N-soliton solutions, and describe a set of N-interacting solitary waves. For instance, an example of a 2-soliton exact solution of the KdV equation is

$$u(x,t) = \frac{12\,[3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)]}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}.$$

corresponding to $k_1 = 2$, $k_2 = 1$. Remarkably it took over seventy years from the discovery of the solitary wave solution of the KdV equation to the realization that it was just the first member of a whole family of exact solutions. Equally remarkably, it is clear that this 2-soliton solution could hardly have been guessed. Figure 3 shows the graph of this solution. Note that as $t \rightarrow \pm\infty$,

$$u(x,t) \rightarrow 8\,\text{sech}^2\,(2x - 32t \pm 2x_1) + 2\,\text{sech}^2\,(x - 4t \pm x_2)$$

where $x_1 = -\ln 3$ and $x_2 = \frac{1}{2}\ln 3$. To verify these expressions, first put $x - 16t = \zeta$ and then take the limit $t \rightarrow \pm\infty$ with ζ fixed: then repeat with $x - 4t = \zeta$. Thus the 2-soliton solution describes the interaction of 2 solitary waves, in which the larger overtakes the smaller, interacts nonlinearly with it (at about t = 0), and then both waves emerge intact. The only remant of the interaction is a phase shift, ddescribed by the constants x_1 and x_2. Here the larger wave has been shifted forward by $\frac{1}{2}\ln 3$ and the smaller wave shifted back by ln 3. A Japanese mathematician Hirota has obtained a simple elegant expression for the N-soliton solution. it is

$$u(x,t) = 2 \frac{\partial^2}{\partial x^2} \ln [\det (I +)P].$$

where I is the N x N identity matrix and P is the N x N matrix whose (n, m) element is

$$\frac{d_n d_m}{(k_n + k_m)} \exp [- (k_n + k_m) x + 4(k_n^3 + k_m^3) t].$$

Here $-k^2_1$, ----, $-k^2_N$ are the N values of λ introduced above and d_1, ---, d_N are certain constants arising from the solution of the ordinary differential equation for ψ.

Thus, following the initial discovery of the soliton by Kruskal and Zabusky, it rapidly became clear from these properties, and other equally interesting results that the soliton was the key ingredient of the KdV equation. However, the true significance of the soliton only became apparent with two further developments. First was the recognition that the KdV equation was not just a model for the propagation of waves on water, but occurred in a wide variety of physical systems, and amongst other applications, described waves in lattices, plasmas, elastic rods, and in the atmosphere and ocean. Further, it was found that the KdV equation is not alone in possessing these remarkable properties, and many other differential equations describing nonlinear waves possess similar properties. The solitary wave has come a long way since Russell's first observation. It is appropriate to conclude this short review with a picture (Figure 4) showing one of the more dramatic naturally occuring solitary waves. It is an atmospheric wave, marked by the line of cloud, moving horizontally on a low-level inversion layer. It occurs often in the Gulf of Carpentaria region of northern Australia, where it is known as the "Morning Glory", since it typically appears early in the morning. The picture was taken at Bourketown and the wave is propagating towards the observer. These waves are often of very large amplitude and have been observed to propagate over large distances. They are a significant component of tropical meso-scale meteorology and in some instances have been identified as potential aviation hazards.

Further reading:

Scott A.C., Chu F.Y.F. and McLaughlin D.W. "The soliton: a new concept in applied science," Proc. IEEE, 61: 1443 - 1483 (1973).

Miura R.M. "The Korteweg-de Vries equation: a survey of resuts." SIAM Review, 18: 412 - 459 (1976)

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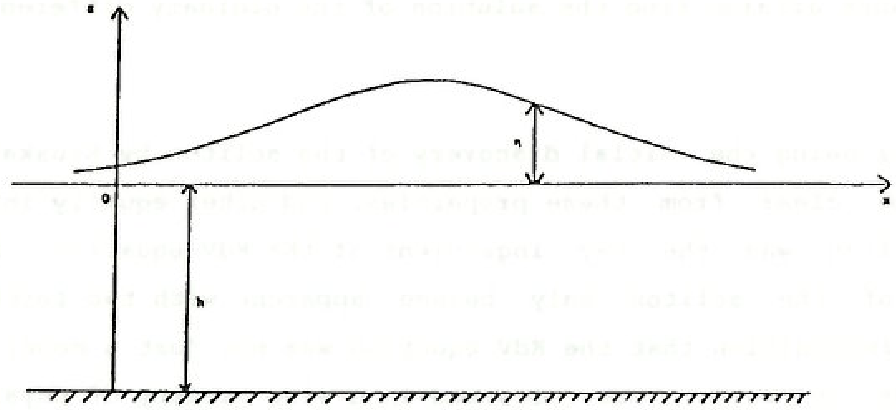

Figure 1: The co-ordinate system. Here z is a vertical co-ordinate, and the fluid has undisturbed depth h. The solitary wave has displacement η.

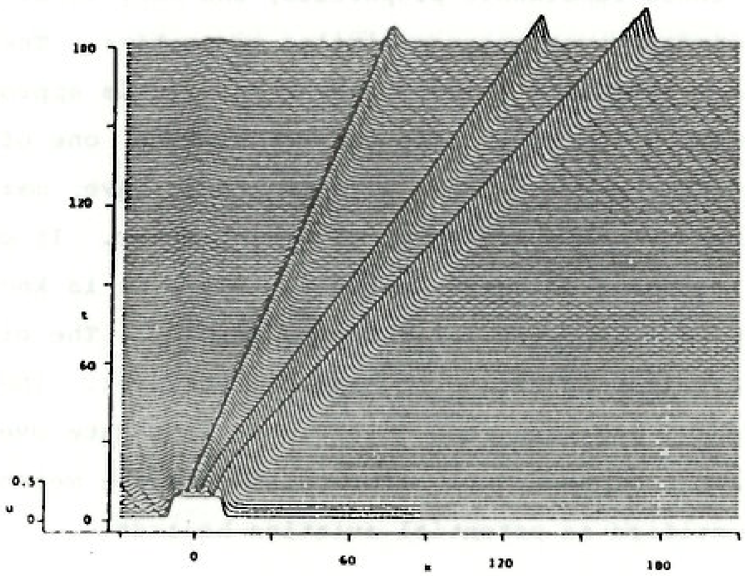

Figure 2: A plot of the evolution of three solitons from the initial condition shown. The plot is obtained by numerical integration of the KdV equation. The very small oscillations seen entering the domain from the right-hand boundary are due to numerical noise. However the oscillations in the bottom left-hand corner are genuine, and are the left-propagating decaying oscillations.

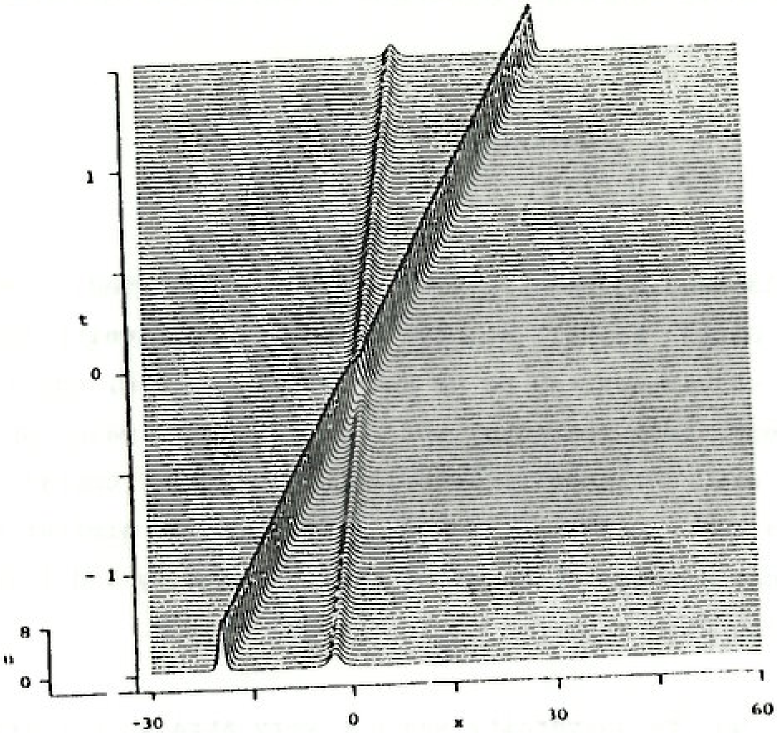

figure 3: A plot of the 2-soliton interaction for the case $k_1 = 1$ and $k_2 = 2$. Note the phase shift near $t = 0$.

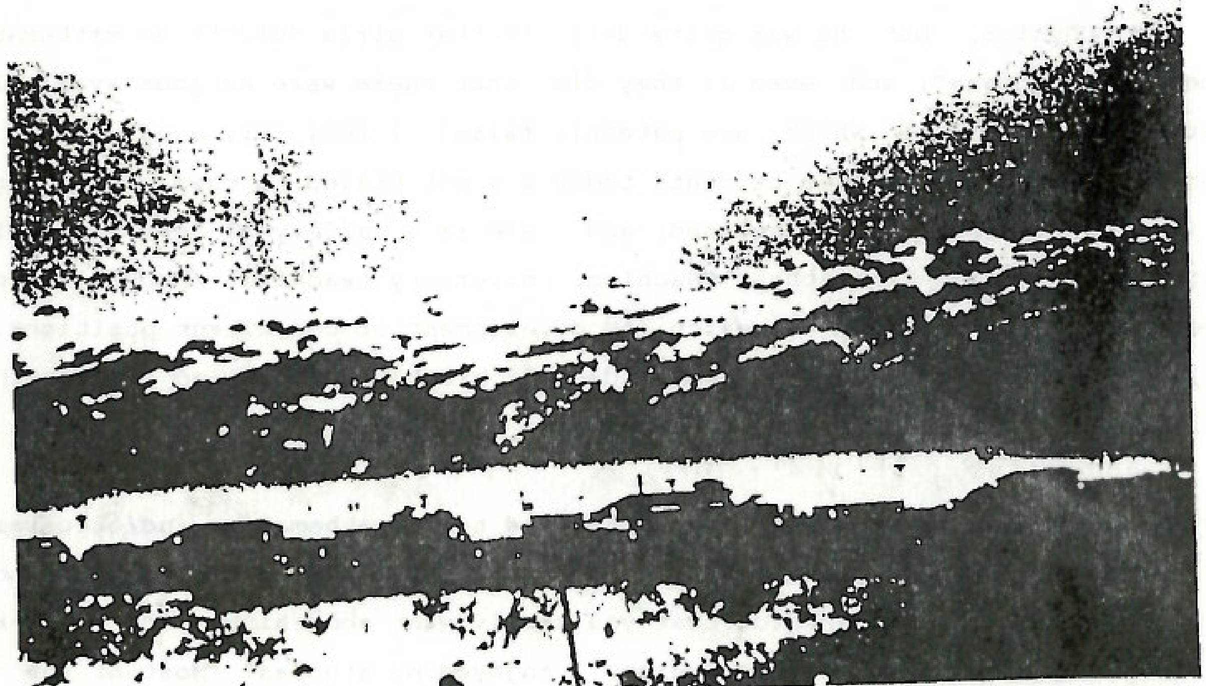

Figure 4: The "Morning Glory" of the Gulf of Carpentaria. The cloud line marks the position of the wave which is propagating towards the observer at about 10m/s