Parabola Volume 23, Issue 2 (1987)

THE 1987 SCHOOL MATHEMATICS COMPETITION

JUNIOR DIVISION

1. In Randwick, the cats, I declare,
   They number one third of a square.
   If a quarter did roam,
   Just a cube would stay home.
   How many, at least, must be there?

Solution: Let x be the number of cats. The verse tells us that $x = \frac{1}{3} y^2$ and $\frac{3}{4} x = z^3$ i.e. $\frac{1}{4} y^2 = z^3$, where y, z are integers. Hence y is divisible by 3 and 2, and so by 6. If we let $y = 6w$, where w is an integer, then $9w^2 = z^3$. The least z satisfying this equation is $z = 9$ (with $w = 9$) and hence $y = 54$ and $x = 972$. So, there must be at least 972 cats.

To find all solutions we proceed as follows. We denote by $a|b$ the statement that a is an exact divisor of b, i.e. b is divisible by a. Now $9w^2 = z^3$ implies $9|z^3$ and so $3|z$ and then $27|z^3$. But then $3|w^2$, so $3|w$ and $81|9w^2$. Now we have $3^4|z^3$, so $3^2|z$ i.e. $z = 9v$ for some integer v and also $w = 9u$ for an integer u such that $u^2 = v^3$. A similar argument shows that v must be a square, say $v = n^2$ for some integer n. Thus we have the general solution $z = 9n^2$, $w = 9n^3$, $y = 54n^3$ and $x = 972n^6$ for any integer n.

2. On a certain car, tyre wear is proportional to distance travelled, front tyres lasting X kilometres and rear tyres lasting Y kilometres. $(X < Y)$. A salesman claims that a set of tyres lasts at least $(X+Y)/2$ kilometres provided you interchange front and rear tyres after an appropriate distance. Investigate.

Solution: Suppose the first pair of tyres is put on the front for z km, then on the rear, and the second pair is on the rear for z km, then on the front. The longest time that the set of 4 tyres will last will occur when both pairs wear out at the same time. The first pair has worn down $\frac{z}{X}$ of the tread when it is swapped, so has $1 - \frac{z}{X}$ of the tread left. This then lasts another $(1 - \frac{z}{X})Y$ km and so the total lifetime for the first pair is $z + (1 - \frac{z}{X})Y$ km.

Similarly the second pair lasts $z + (1 - \frac{z}{Y})X$ km.

For maximum life $z + (1 - \frac{z}{X})Y = z + (1 - \frac{z}{Y})X$ i.e. $z(\frac{X}{Y} - \frac{Y}{X}) = X - Y$

$$\text{or} \quad z = \frac{(X - Y)XY}{X^2 - Y^2} = \frac{XY}{X+Y} \text{ km} .$$

With this time on the front, the total lifetime $= \frac{XY}{X+Y} + (1 - \frac{Y}{X+Y})Y = \frac{2XY}{X+Y}$.

Since $(X+Y)^2 - 4XY = X^2 - 2XY + Y^2 = (X-Y)^2 \geq 0$ we have $(X+Y)^2 \geq 4XY$

or $X+Y \geq 2\sqrt{XY}$ (This inequality is the arithmetic/geometric mean inequality $\frac{1}{2}(X+Y) \geq (XY)^{1/2}$.)

Putting this inequality in to the above gives a longest life

$$\leq \frac{\frac{1}{2}(X+Y)^2}{X + Y} = \frac{1}{2}(X+Y).$$

The salesman's claim is not true, unless $\frac{1}{2}(X+Y) = \sqrt{XY}$ which only occurs if $X = Y$ (i.e. tyre wear is even, front and back) and this is not allowed.

Consequently we do not trust the car salesman!

3. We place in a box, 1987 white marbles and 7891 black marbles. We also have 1000 black marbles outside the box. We remove two marbles from the box. If they have different colours, we put the white one back in the box. If they have the same colour, we put a black marble into the box. We continue doing this until only one marble is left. What is its colour?

<u>Solution</u>: Two white marbles (at most) must be drawn out for each black marble put in, so we have sufficient black marbles.

At each step, the total number of marbles in the box decreases by 1, so we must eventually end up with only a single marble.

The number of white marbles either stays constant or reduces by 2 at each step. Since we start with an odd number of white marbles, we must finish with an odd number of white marbles - so the last marble is white!

4. Prove that the angle at vertex C of triangle ABC is a right angle if and only if

$$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2},$$

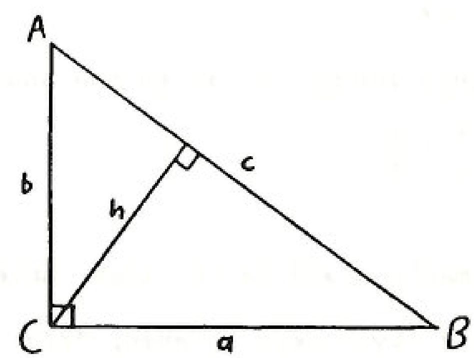
where a, b are the lengths of the two sides from vertex C and h is the shortest distance from C to the longest side AB.

Solution: Let AB be c and suppose angle at C is a right angle. The area of the triangle ABC $= \frac{1}{2}ab$ and $= \frac{1}{2}ch$.

Hence $ab = ch$, so $a^2b^2 = c^2h^2$.

By Pythagoras $c^2 = a^2 + b^2$, so we get $a^2b^2 = (a^2+b^2)h^2$

i.e. $\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2}$.



Conversely, suppose that

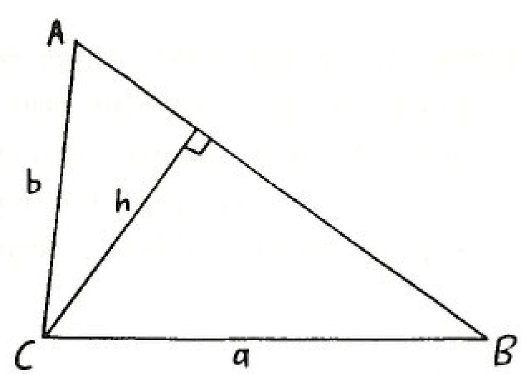
$$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Since $\sin A = \frac{h}{b}$, $\sin B = \frac{h}{a}$ we get

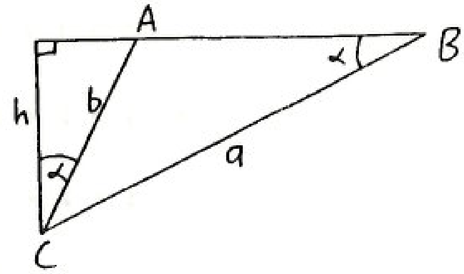
$$\sin^2 A + \sin^2 B = h^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 1.$$

But $\cos^2 B + \sin^2 B = 1$, so $\sin^2 A = \cos^2 B$. As AB is the largest side, both A and B are acute angles and so $\sin A = \cos B$ and $A = 90^\circ - B$.

Thus $C = 90^\circ$ as required.



Notice that the result is no longer true if we do not assume AB is the longest side. Consider the special case shown, where $h = a \sin \alpha = b \cos \alpha$.



5. Prove that if a, b, c are integers and the odd number n is a divisor of $a + b + c$ and $a^2 + b^2 + c^2$ then it is also a divisor of $a^4 + b^4 + c^4$. Is this still true if n is even?

<u>Solution</u>: We show that the result is true for any n, odd or even. (Recall the notation of question 1.)

If $n | a + b + c$ then $n | (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$.

If also $n | a^2 + b^2 + c^2$, then $n | 2(ab + bc + ca)$ and further,

$n | 2(ab + bc + ca)^2 = 2(a^2 b^2 + b^2 c^2 + c^2 a^2) + 4\, abc(a + b + c)$.

Since $n | a + b + c$, we get $n | 2(a^2 b^2 + b^2 c^2 + c^2 a^2)$.

Finally, $n | (a^2 + b^2 + c^2)^2 = a^4 + b^4 + c^4 + 2(a^2 b^2 + b^2 c^2 + c^2 a^2)$ as well,

and so $n | a^4 + b^4 + c^4$.

6. How many positive integers less than 1001 can be expressed in the form $[2x] + [4x] + [6x] + [8x]$, where x is some real number? ($[x]$ denotes the greatest integer less than or equal to x).

<u>Solution</u>: Write $x = n + \theta$ where $0 \le \theta < 1$.

Then $[2x] + [4x] + [6x] + [8x] = 20n + [2\theta] + [4\theta] + [6\theta] + [8\theta]$.

$$\begin{aligned} \text{Now } [2\theta] + [4\theta] + [6\theta] + [8\theta] &= 0 + 0 + 0 + 1 = 1 \quad \text{if } \tfrac{1}{8} \le \theta < \tfrac{1}{6} \\ &= 0 + 0 + 1 + 1 = 2 \quad \text{if } \tfrac{1}{6} \le \theta < \tfrac{1}{4} \\ &= 0 + 1 + 1 + 2 = 4 \quad \text{if } \tfrac{1}{4} \le \theta < \tfrac{1}{3} \\ &= 0 + 1 + 2 + 2 = 5 \quad \text{if } \tfrac{1}{3} \le \theta < \tfrac{3}{8} \end{aligned}$$

$$= 0 + 1 + 2 + 3 = 6 \quad \text{if } \frac{3}{8} \le \theta < \frac{1}{2}$$

$$= 1 + 2 + 3 + 4 = 10 \quad \text{if } \frac{1}{2} \le \theta < \frac{5}{8}$$

$$= 1 + 2 + 3 + 5 = 11 \quad \text{if } \frac{5}{8} \le \theta < \frac{2}{3}$$

$$= 1 + 2 + 4 + 5 = 12 \quad \text{if } \frac{2}{3} \le \theta < \frac{3}{4}$$

$$= 1 + 3 + 4 + 6 = 14 \quad \text{if } \frac{3}{4} \le \theta < \frac{2}{3}$$

$$= 1 + 3 + 5 + 6 = 15 \quad \text{if } \frac{2}{3} \le \theta < \frac{7}{8}$$

$$= 1 + 3 + 5 + 7 = 16 \quad \text{if } \frac{7}{8} \le \theta < 1.$$

So, by suitable choice of n and θ, i.e. x, any number can be obtained which does not end in digit 3,7,8 or 9.

Among the numbers 1 to 1000, 100 numbers end in each of 3,7,8,9 so 600 numbers can be obtained in the required form.

An alternative and prettier proof.

Write $y = 24x$ and $f(y) = [\frac{1}{12} y] + [\frac{1}{6} y] + [\frac{1}{4} y] + [\frac{1}{3} y]$.

As y increases from 0, the function $f(y)$ remains constant, except for jumps which occur each time y is an integer multiple of 3 or 4. Now $f(y) = 1000$ when $y = 1200$.

There are 400 multiples of 3 and 300 multiples of 4 - but we must not count the 100 multiples of 12 twice! So the number of different values taken by f for $0 \le y \le 1200$ is $400 + 300 - 100 = 600$.

7. Given a number p, the number q is obtained by writing p in binary notation and interpreting the result as a number written in base 5. For example, $p = 13 = 2^3 + 2^2 + 1$ gives $q = 5^3 + 5^2 + 1 = 151$. Find p such that $q = 65p$.

<u>Solution</u>: Let $p = x_0 + x_1 2 + x_2 2^2 + x_3 2^3 + x_4 2^4 + x_5 2^5 + x_6 2^6 + \ldots\ldots$

then $q = x_0 + x_1 5 + x_2 5^2 + x_3 5^3 + x_4 5^4 + x_5 5^5 + x_6 5^6 + \ldots$

where each x_i is 0 or 1, and the sums terminate. Since in our problem q 65p then $x_0 = 0$ as q must be divisible by 5 Dividing by 5 we have

$$p = 2x_1 + 2^2x_2 + 2^3x_3 + 2^4x_4 + 2^5x_5 + 2^6x_6 + \ldots$$

$$q = 13p = x_1 + 5x_2 + 5^2x_3 + 5^3x_4 + 5^4x_5 + 5^5x_6 + \ldots$$

Since $13.2^4 = 208$, $13.2^5 = 416$, $13.2^6 = 832$ while $5^3 = 125$, $5^4 = 625$, $5^5 = 3125$ a consideration of sizes shows that $x_5 \Leftrightarrow 0$ i.e. $x_5 = 1$, while $x_6 = x_7 = \ldots = 0$.

So $q = x_1 + 5x_2 + 25x_3 + 125x_4 + 625$ and is divisible by 13.

Since $25 = 2.13 - 1$, $125 = 10.13-5$ and $625 = 48.13 + 1$
we get that $13 | x_1 + 5x_2 - x_3 - 5x_4 + 1$.

Since x_1, x_2, x_3, x_4 are 0 or 1, this forces the values $x_1 = 0$, $x_3 = 1$, $x_2 = x_4$ and then by trial and error $x_2 = x_4 = 1$ rather than $x_2 = x_4 = 0$.

Hence $p = 0 + 0.2 + 1.2^2 + 1.2^3 + 1.2^4 + 1.2^5 = 60$.

An alternative proof would be to write out the corresponding p, q and 65p for each of $p = 2^0, 2^1, 2^2, 2^3, 2^4, 2^5$ and by trial and error, see that
$65(2^2 + 2^3 + 2^4 + 2^5) = 5^2 + 5^3 + 5^4 + 5^5$.

SENIOR DIVISION

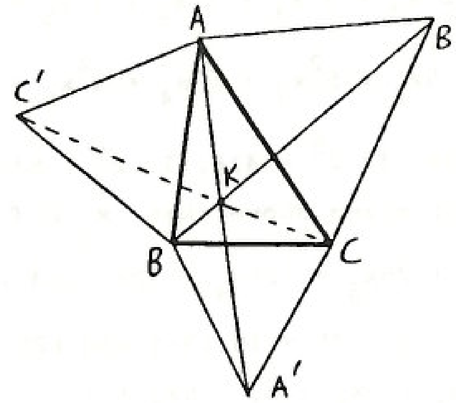
1. We place in a box, 1987 white marbles and 7891 black marbles. We also have 1000 black marbles outside the box. We remove two marbles from the box. If they have different colours, we put the white one back in the box. If they have the same colour, we put a black marble into the box. We continue doing this until only one marble is left. What is its colour?

Solution: As for Junior question 3.

2. Equilateral triangles A'BC, B'CA, C'AB are drawn external to the triangle ABC. Show that AA', BB', CC' are equal and that they intersect in a common point.

Solution: Triangles A'BC, AB'C
and ABC' are each equilateral

Suppose AA′ and BB′ intersect at K.



Now A′C = BC and CA = CB′ and /A′CA = /BCB′ so triangles A′CA and BCB′ are congruent (SAS) - (1). Hence AA′ = BB′.

Similarly triangles ABA′ and C′BC are congruent so AA′ = CC′ and we have AA′ = BB′ = CC′.

To show that AA′, BB′ and CC′ intersect in a common point we show that if we join CK and C′K then all the angles at K are 60° and hence C′KC is a straight line, i.e. CC′ passes through the intersection of AA′, BB′.
By (1), /KAC = /KB′C so KAB′C is a cyclic quadrilateral (equal angles on a chord) and hence /AKB′ = /ACB′ = 60° and /CKB′ = /CAB′ = 60°. Similarly KBA′C is cyclic and /BKA′ = /BCA′ = 60° and /A′KC = /A′BC = 60°. Since angles at K sum to 360°, /AKB = 120°.
Now /AC′B = 60° so AC′BK is cyclic (opposite angles supplementary). Hence /C′KA = /C′BA = 60° and /C′KB = /C′AB = 60° and all angles at K are 60°.

3. How many positive integers less than 1987 can be expressed in the form [2x] + [4x] + [6x] + [8x], where x is some real number? ([x] denotes the greatest integer less than or equal to x).

Solution: As for Junior question 6, any number not ending in 3,7,8 or 9 can be so expresssed. There are 199 numbers < 1987 which end in each of 1,2,4,5,6 and 198 ending in 0, so in total 1193 numbers.

Using the alternative method, f(2386) = 1987 so the answer is

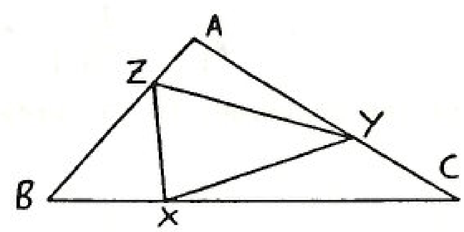
$$[\frac{2386}{3}] + [\frac{2386}{4}] - [\frac{2386}{12}] = 1193.$$

4. 18 football teams play against each other in a 17 round competition. Prove that after the 8th round there are three teams such that no two among them have played each other.

Solution: By renumbering teams if needed, we may suppose that team 1 has played teams 2 through 9 and has not played teams 10 through 18.
If any pair of teams in {10,11, ..., 18} have not played each other, the three teams, this pair and 1, provides a solution to the problem. The above will always happen, unless all pairs of teams in {10,11, ..., 18} have already played each other. Since there are 9 teams and 8 rounds, then these teams must have played in their own "sub-league" (i.e. against one another and not against any of {1,2, ..., 9}) in every one of the 8 rounds. However this is impossible as there are an odd number of them in the "sub-league".

5. Let ABC be a triangle and X,Y,Z points on the sides BC, CA, AB respectively. Show that if $BX \le XC$, $CY \le YA$, $AZ \le ZB$, then the area of triangle XYZ is not less than one quarter of the area of triangle ABC. Show also that, in any case, one of the corner triangles AZY, BXZ, CYZ has area not greater than the area of triangle XYZ.

Solution: Suppose $BX \le XC$, $CY \le YA$, $AZ \le ZB$.

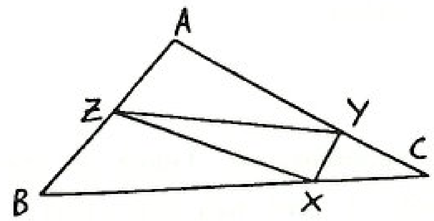


If we leave Y, Z fixed and move X towards the midpoint of BC, then in triangle XYZ, the base YZ remains the same, but the altitude from X to YZ decreases or (if Y,Z are midpoints of AC, AB) stays the same. Hence the area of triangle XYZ decreases (or stays the same) as X is moved to the midpoint of BC. A similar argument applies to Y and Z. Hence area of triangle XYZ is greater than or equal to the area of the triangle formed by joining the midpoints of the sides and this has area one quarter that of triangle ABC.

Also the three corner triangles have total area $\le \frac{3}{4}$ area of triangle ABC, so one of them must have area $\le \frac{1}{4}$ area of triangle ABC and so $\le$ area of triangle XYZ.

All other positions of X, Y, Z give cases equivalent to the above, or to the case where

$BX > XC$ and $CY \le YA$.



In this case the altitude from C to XY is less than the altitude from Z to XY and hence area of triangle $XYC <$ area of triangle XYZ.

6. Let $x_{n+1} = x_n + x_n^2$, and $x_1 = 1/3$. Find $\sum_{n=1}^{\infty} \frac{1}{1+x_n}$.

Solution: We have $x_{n+1} = x_n + x_n^2 = x_n(1 + x_n)$,

so $$\frac{1}{1+x_n} = \frac{x_n}{x_{n+1}} = \frac{x_n^2}{x_n x_{n+1}} = \frac{x_{n+1} - x_n}{x_n x_{n+1}} = \frac{1}{x_n} - \frac{1}{x_{n+1}}.$$

Then $$\sum_{n=1}^{N} \frac{1}{1+x_n} = \left(\frac{1}{x_1} - \frac{1}{x_2}\right) + \left(\frac{1}{x_2} - \frac{1}{x_3}\right) + \dots + \left(\frac{1}{x_N} - \frac{1}{x_{N+1}}\right)$$

$$= \frac{1}{x_1} - \frac{1}{x_{N+1}} = 3 - \frac{1}{x_{N+1}}.$$

Thus the series converges to 3 provided $x_{N+1} \to \infty$ as $N \to \infty$.

The first few terms are $x_1 = \frac{1}{3}$, $x_2 = \frac{4}{9}$, $x_3 = \frac{52}{81}$, $x_4 = 1.054$,

and from then on $x_{n+1} = x_n + x_n^2 > x_n + 1 > n-2$ $(n \ge 4)$ so $x_n \to \infty$ as $n \to \infty$.

7. Find a solution of

$$\frac{2}{1987} = \frac{1}{x} + \frac{1}{y},$$

where x, y are integers such that $0 < x < y$. Show that there are no other solutions of this kind.

Solution: By testing for divisibility of 1987 by all primes less than $\sqrt{1987} = 44.5$, namely 2,3,5,7,11,13,17,19,23,29,31,37,41,43, we find that 1987 is prime.

We will prove that the only solution of

$$\frac{2}{p} = \frac{1}{x} + \frac{1}{y}, \quad 0 < x < y \text{ when } p \text{ is an odd prime}$$

is $x = \frac{1}{2}(p+1)$, $y = \frac{1}{2}p(p+1)$.

(so the solution with $p = 1987$ is $x = 994$, $y = 1,975,078$)

Since $\frac{2}{p} = \frac{1}{x} + \frac{1}{y}$ and $\frac{1}{x} > \frac{1}{y}$ we see that $\frac{1}{p} < \frac{1}{x}$ i.e. that $x < p$.

Now $2xy = p(x+y)$ so that $p|y$ and we can write $y = pz$ with z an integer.

Now $2xz = x + pz$ so $(2z - 1)x = pz$. Since z, $2z-1$ have no common factors, and x, p have no common factors, then $x = z$ and $2z-1 = p$.

Thus $x = \frac{1}{2}(p+1)$ and $y = px$.

As an exercise you might like to try to find the general solution of

$$\frac{2}{n} = \frac{1}{x} + \frac{1}{y} \text{ with } x, y \text{ positive integers,}$$

where n is an odd positive integer, not necessarily prime. The answer is:

$x = \frac{(a+b)n}{2a}$, $y = \frac{(a+b)n}{2b}$, where a and b are any two divisors of n which have no common factors.