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HOW TO CALCULATE COSINES

By Bill McKee *

Have you ever wondered how your calculator or computer finds quantities such as $\sin(63^\circ)$ or $\tan(17^\circ)$? These trigonometrical quantities are first introduced to us in terms of the ratios of the sides of triangles. What is (almost) certain is that our calculators do not contain a little gnome who constructs triangles, measures the ratios of the sides and then tells us what we wanted to know. How then do our calculators find the values of trigonometrical functions? The first thing to realise is that our calculators only give us approximate values. This is illustrated by the simple example of $\sin(60^\circ)$ which, as everybody knows, has the value $\sqrt{3}/2$. Now the decimal representation of $\sqrt{3}$, and hence of $\sqrt{3}/2$, does not terminate, i.e. it goes on for ever; the first part of it being $\sqrt{3}/2 = 0.86602540378444\ldots$

My calculator gives $\sin(60^\circ) = 0.866025403$ and so is only giving me an approximation to $\sin(60^\circ)$. For almost all practical purposes, this is more accuracy than is required; four or five decimal places are usually sufficient. As a matter of principle, the approximations to trigonometrical functions, and other functions such as logarithms, given by computers and calculators should be correct to the number of decimals displayed and my calculator is wrong in the last decimal place since the last figure should have been rounded up to 4. How then do calculators find these approximations?

We will illustrate the principles which underlie the methods used by considering the cosine function $\cos\theta$ where θ is an angle measured in degrees and $0 \le \theta \le 90$. You should be able to show quite easily that, if we know $\cos\theta$ for $0 \le \theta \le \pi/2$, we can readily find $\cos\theta$ for any other angle θ. Now, degrees are not the most natural ways in which to represent angles because we humans arbitrarily decide to use this measuring system which assign 90 degrees to one right angle. Far more fundamental is radian measure which assigns $\pi/2$ radians to one right angle.

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Thus, if an angle is θ degrees, its value in radians is x where $x = \pi \theta / 180$. For those of you studying calculus expressing things in radians makes derivatives simple, e.g. $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\sin x) = \cos x$.

From now on, we will use radians and consider cos x. The basic idea used in approximating cos x is that cos x can be shown to have an infinite series expansion

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{1}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

We recall that, for any positive integer n, the factorial n! is defined by

$$n! = n(n-1) \, .. \, 3.2.1 \, ,$$

i.e., it is the product of all the positive integers less than or equal to n. It is also convenient to define

$$0! = 1.$$

The series (1) is an infinite series, i.e., it goes on for ever. It would take a computer or calculator an infinite time to sum the series because each operation takes a finite time to perform. We cannot afford to wait that long and so the best we can do is chop (1) off after a finite number of terms and so find

$$S_N(x) = \sum_{n=0}^{N} \frac{(-1)^n x^{2n}}{(2n)!} \tag{2}$$

$$= 1 - \frac{x^2}{2!} + \, .. \qquad + \frac{(-1)^N x^{2N}}{(2N)!} \, ,$$

for some integer N. It can be shown that for any given x we can make $S_N(x)$ as close as we like to cos x by taking a sufficiently large number of terms in (2), that is, by making N large enough. In fact, some fancy manipulation can be used to show that the error in (2) is less in magnitude than the first neglected term, that is,

$$-\frac{x^{2N+2}}{(2N+2)!} < \cos x - S_N(x) < \frac{x^{2N+2}}{(2N+2)!}$$

for $0 \le x \le \pi/2$.

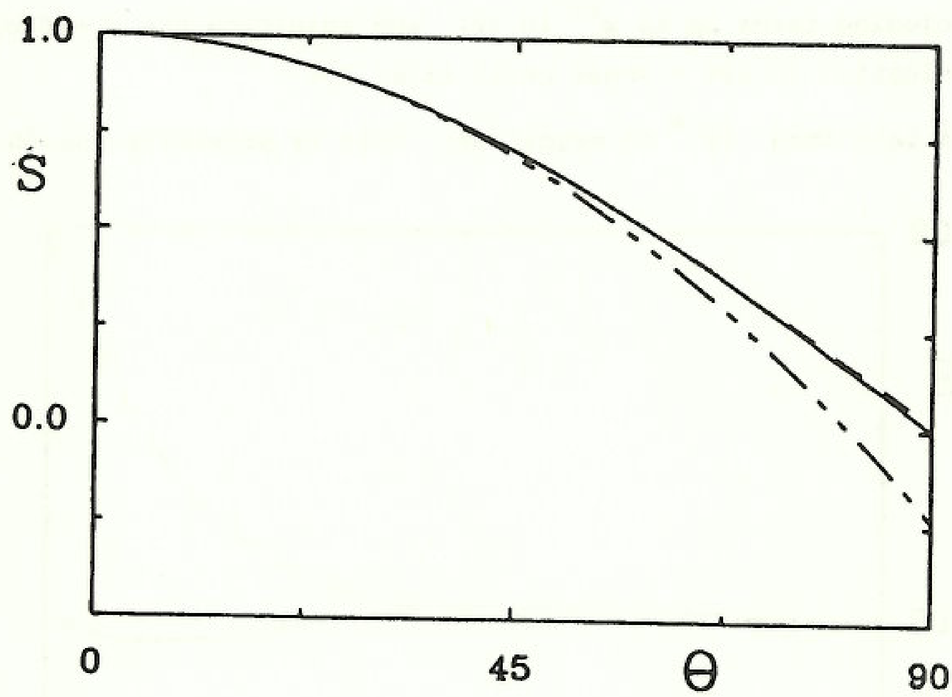
This tells us how many terms to take in order to get any specified accuracy. It would be wasteful to use more terms than are needed. As an exercise, you might like to work out the smallest number of terms in (2) that would be sufficient to find cos (1) correct to 7 decimal places.

Well, how good is the series (2)? In the first figure we show graphs of cos x and

$$S_1(x) = 1 - \frac{x^2}{2!} ,$$

$$S_2(x) = S_1(x) + \frac{x^4}{4!}$$

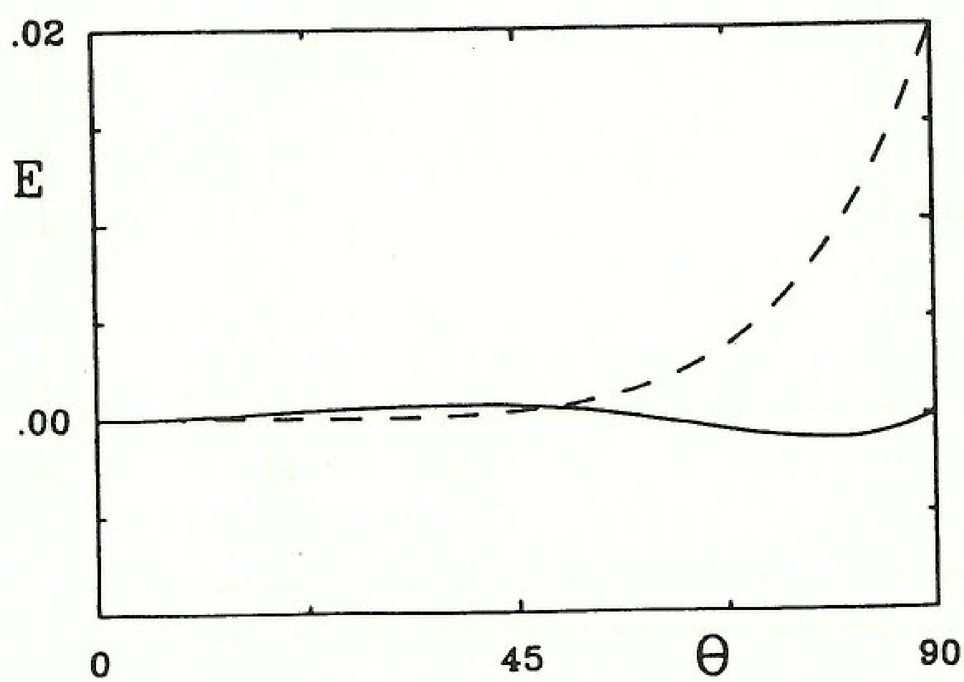
and $$S_3(x) = S_2(x) - \frac{x^6}{6!} .$$



The solid line is cos x and is correct to about 14 decimal places. The lower dotted line is $S_1(x)$. You might be able to distinguish the upper dotted line which is $S_2(x)$ just above the solid line. The third dotted line $S_3(x)$ is indistinguishable from cos x on the graph. This does not mean that $S_3(x) = \cos x$ but merely that our eyes cannot tell them apart on the scale of the graph. Note also that the scale on the abscissa is expressed in terms of degrees $\theta = 180\ x/\pi$.

You will also notice that the errors in our simple approximations $S_1(x)$ and $S_2(x)$ are very small near $x = 0$ but increase as x increases. In fact we can get a better overall approximation for $0 \leq x \leq \pi/2$ by fiddling a little with the coefficients. We will illustrate this fact by considering $S_2(x)$. If we adjust the coefficient of x^2 from $-\frac{1}{2!} = -0.5$ to -0.49670 and that of x^4 from $\frac{1}{24.!} = 0.04166$ .. (repeating) to 0.03705 we get the approximation

$T_2(x) = 1 - 0.49670x^2 + 0.03705x^4$ which is not quite so accurate near $x = 0$ but is much better for larger values of x and so gives a better overall approximation to $\cos x$ for $0 \leq x \leq \pi/2$. This is illustrated in the second figure which shows the error in $S_2(x)$ (i.e. $S_2(x) - \cos x$) by a dotted line and the error in $T_2(x)$ by a solid line. Again the abscissa is expressed in degrees θ.Taking $S_5(x)$, i.e., including terms up to x^{10} in (2) and adjusting the coefficients slightly gives an approximation to cos x whose worst error for $0 \leq x \leq \pi/2$ is less than 10^{-8} in magnitude. This is generally enough accuracy.



Well, we have seen how to calculate cos x approximately. What about the other trigonometrical functions? You might like to experiment with the series for sin x

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= x - x^3/3! + x^5/5! - \ldots$$

The series for tan is a little nastier since there is no simple expression for the general term. The first few terms are given by

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$$

Remember that in all of these x is expressed in radians. Happy computing!

Editor's footnote. Trigonometry goes back a long time. For example the Greek astronomer Hipparchus of Nicaea (c 180 - c125BC) who is often called "the father of trigonometry" was probably the first to compile trigonometric tables. These tables listed the corresponding values of arc ADB and chord ACB for a series of angles. Earlier Aristarchus of Samos (c 310 - 230BC) had used trigonometry to estimate, for example, the relative distances of the sun and moon from the earth. Later the ancient world's greatest astonomer Ptolemy of Alexandria (who was alive in the 1st half of the 1st century AD) published a "book" which later became known to us as Almagest (from the Arabic meaning "the greatest"). This work was the standard authority on astronomy/trigonometry for nearly 1500 years (and in fact Copernicus largely relied on it even when he was overthrowing Ptolemy's geocentric theory of the solar system). With the decline of the old Greek civilization it was the Hindus and especially the Arabs who kept mathematics alive. Trigonometry was useful to both these peoples for religious reasons (astrology, finding the direction of Mecca etc), and the Hindus started to work with the half chord AC which is exactly our sine if the circle has radius one. Eventually (about 1200AD) the Arab mathematical libraries were translated into Latin for study in Christian Europe. Although a number of early European mathematicians studied trigonometry it was not until 1748 that Euler introduced dimensionless trig functions by assuming the radius of the circle was one. He also gave us the series expansions that Bill has used.