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SOLUTIONS OF PROBLEMS Q. 708 - Q. 719.

Q.708. A four digit number a b c d has the property that $a + b = c \times d$ and also $a \times b = c + d$. Find all possibilities.

ANSWER. Let x, y be two positive integers with $x \geq y$. If $y \geq 2$, then $xy \geq 2x \geq x + y$. In this we have equality, $xy = x + y$, if and only if $x = y = 2$ Hence if $x y < x + y$ then $y = 1$.

Now let $a + b = c \times d$ and $a \times b = c + d$.

One of the following three situations must apply.

(I) $a \times b = a + b$ and $c \times d = c + d$

(II) $a \times b > a + b$ and $c \times d < c + d$

(III) $a \times b < a + b$ and $c \times d > c + d$

If (I) applies, the previous discussion shows that $a = b = c = d = 2$.

If (II) applies, one of c, d is 1, and we must have $a \times b = a + b + 1$.

$$\therefore \quad a = \frac{b+1}{b-1} .$$

If b exceeds 3 it is easy to see that $1 < \frac{b+1}{b-1} < 2$. Since a is a whole number there are only two possibilities; i.e. $b = 2$, $a = 3$; or $b = 3$, $a = 2$. Now we have $c \times d = a + b = 5$. Hence c, d are 1 and 5 in either order.

If (III) applies, we have a similar result, interchanging a, b with c, d.

Thus the only possible numbers abcd are {2222, 2315, 2351, 3215, 3251, 1523, 1532, 5123, 5123}.

Q. 709. Prove that the numbers 49, 4489, 444889, ... (each obtained by inserting 48 into the middle of the preceding number) are all perfect squares.

ANSWER. $\overset{n\text{ digits}}{44\,.....\,4}\overset{(n-1)\text{ digits}}{88\,.........\,8}9 = \overset{2n\text{ digits}}{44\,........\,4} + \overset{n\text{ digits}}{44\,.......\,4} + 1$

$$= \frac{4}{9}(10^{2n} - 1) + \frac{4}{9}(10^{n} - 1) + 1.$$

$$= \frac{4 \times (10^{n})^{2} + 4 \times 10^{n} + (9 - 4 - 4)}{9}$$

$$= \left(\frac{2 \times 10^{n} + 1}{3} \right)^{2}$$

Hence for any n, this number is a perfect square. (Note that the numerator 2000 ... 01 is exactly divisible by 3 since the sum of the digits is 3).

Q.710. A number is "palindromic" if it reads the same with the digits in reverse order (e.g. 42724). Find all palindromic numbers with 6 digits which are obtainable by adding two palindromic 5 - digit numbers.

ANSWER.

$$\begin{array}{rr} & a_1 b_1 c\ b_2 a_2 \\ + & \underline{A_1 B_1 C\ B_2 A_2} \\ & u_1 v_1 w_1 w_2 v_2 u_2 \end{array}$$

Clearly $u_1 = 1$. $\therefore$ $u_2 = 1$ and $a_2 + A_2 = 11$.

$\therefore$ v_1 must be either 1 or 2

Case 1 $v_1 = 1$. then $b_1 + B_1 < 10$, since no 1 is carried.

$\therefore$ $1 + b_2 + B_2 = v_2 = 1 \rightarrow b_2 = B_2 = 0 \rightarrow b_1 = B_1 = 0$.

We must have $w_1 = 0$ or 1.

Hence, the only possible answers with $v = 1$ are

110011 or 111111.

(These are obtainable; e.g. 50005 + 60006; 50705 + 60406).

Case 2. $v = 2$. Then $b_1 + B_1 > 10$, and $b_2 + B_2 + 1 = 12$.

Then $w_1 = 1$ or 2

The only possible answers are 121121 and 122221.

[e.g. 57075 + 64046; 57875 + 64346)

Hence there are four possible answers:-

110011; 111111; 121121; 122221.

Q.711. One obvious solution of the equation $(1.001)^x = 1 + x^{1000}$ is $x = 0$. There is another (large) value of x which satisfies the equation. Can you find an approximate value of the second solution?

ANSWER. Since the large solution must obviously be greater than 1000, the term 1 on the R.H.S. is totally negligible compared with x^{1000} and may be omitted when obtaining the approximate solution. Taking natural logarithms

$$(1.001)^x = x^{1000}$$

$$x \ln 1.001 = 1000 \ln x.$$

$$x = A \ln x \text{ where } A = \frac{1000}{\ln 1.001} = 1000500$$

Put $x = A\,y$. We obtain

$$y = \ln A + \ln y = 13.81601 + \ln y$$

Clearly y must be greater than $y_1 = 13.81601$ and indeed therefore somewhat greater than $y_2 = 13.81601 + \ln 13.81601 = 16.441$ ..

We can continue this iterative process, finding successively better approximations to y, by taking $y_n = 13.81601 + \ln y_{n-1}$.

The next few values, $y_3, \ldots, y_6$ are 16.62637, 16.62100, 16.62704, 16.62704,....

Hence a solution of $y = \ln A + \ln y$, to about 7 significant figures, is 16.62704.

$$\therefore\ x = Ay = 1000500 \times 16.62704\ldots$$

$$= 16635347\ .$$

Q. 712. Given positive integers $a\ (\neq 1)$, m, and n, where m and n have no common factor except 1, find the highest common factor of $a^m - 1$ and $a^n - 1$ and prove your result.

ANSWER. Note that $a^m - 1 = (a - 1)(a^{m-1} + a^{m-2} + \ldots + 1)$ and similarly $a - 1$ is a factor of $a^n - 1$. We shall prove that $a - 1$ is in fact the highest common factor of $a^m - 1$ and $a^n - 1$. Let us assume that $m > n > 1$. Then any common factor, c, of $a^m - 1$ and $a^n - 1$ is also a factor of

$$(a^m - 1) - a^{m-n}(a^n - 1) = a^{m-n} - 1.$$

Note that n and m-n have no common factor except 1. (If $p \mid n$ and $p \mid m-n$ then $p \mid (m-n) + n$ i.e. p is a common factor of m and n. $\therefore$ $p = 1$). Thus the h.c.f. of $a^m - 1$ and $a^n - 1$ is also the h.c.f. of $a^{m_1} - 1$ and $a^n - 1$, where $m_1 = m - n < m$, and m_1 and n are still relatively prime.

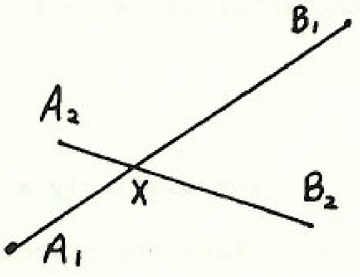
This process can be repeated, at each stage obtaining a pair of numbers $a^x - 1$ and $a^y - 1$ having the h.c.f of $a^m - 1$ and $a^n - 1$ as a common factor, with x and y relatively prime positive integers. The process obviously has to terminate eventually, and it is clear that it does so only when x and y are both equal to 1. Thus the h.c.f. of $a^m - 1$ and $a^m - 1$ is also a factor of $a - 1$, and our assertion is proved.

Q. 713. You are given 100 point in a plane, no three of which are collinear. Any 50 are chosen and labelled A , the others being labelled B. Show that it is possible to draw 50 straight line segments each linking a point labelled A to a point labelled B in such a way that

(i) each of the 100 points is an end point of one line segment.

and (ii) no two line segments intersect.

ANSWER. Let us ignore (ii) for the moment. Then there is only a finite number of ways, N, say, of drawing 50 line sigments using all the 100 given points as end points. (You may be able to find a formula for N, but this is not of importance). For each of these N possible constructions, calculate L, the sum of the lengths of all 50 line segments. Of the N values of L, a minimum answer L_{min} must be obtained. Take any construction for which $L = L_{min}$. We claim that for this construction (ii) is satisfied.



Suppose on the contrary that in our construction there is a pair of the line segments $A_1 B_1$ and $A_2 B_2$ which intersect at X. Then $A_1X + X B_2 > A_1 B_2$ and $A_2 X + X B_2 > A_2 B_1$.

Thus if in our construction we replace these two line segments with $A_1 B_2$ and $A_2 B_1$ we would have another construction satisfying (i) with $L < L_{min}$, a contradiction.

Q. 714. I have four infinite sets of whole numbers X_1, X_2, X_3 and X_4. Every non-negative integer m can be expressed uniquely in the form

$$m = x_1 + x_2 + x_3 + x_4$$

where $x_1 \varepsilon X_1$, $x_2 \varepsilon X_2$, $x_3 \varepsilon X_3$, and $x_4 \varepsilon X_4$.

(i) Show that 0 is in every set, but that no other number can be in more than one set.

(ii) For any $N > 0$ let $n(N)$ be the number of elements of X_1 X_2 X_3 X_4 which are less than N.

Show that $n(N) \geq 4 N^{1/4} - 3$ for any N.

(iii) Find four sets having the stated properties, and if possible a number, N for which

$$n(N) = 4 N^{1/4} - 3.$$

ANSWER. (i) Obviously, since 0 is to be expressible in the form $0 = x_1 + x_2 + x_3 + x_4$ with each $x_i \geq 0$, we must have

$x_1 = x_2 = x_3 = x_4 = 0$. $\therefore$ 0 is in every set X_i.

If any other non-negative integer n occurred say in X_1 and X_2 then $n = n + 0 + 0 + 0 = 0 + n + 0 + 0$ gives two different ways of expressing n in the form $x_1 + x_2 + x_3 + x_4$, contradicting the uniqueness stipulation in the data.

(ii) Let X_1, X_2, X_3, X_4 contain respectively n_1, n_2, n_3, and n_4 numbers less than N. Then (since 0 is counted 4 times on the LHS)

$$n_1 + n_2 + n_3 + n_4 = n(N) + 3.$$

Thus the arithmetic mean of $\{n_1, n_2, n_3, n_4\}$ is $\frac{n(N)+3}{4}$.

Each of the N numbers 0, 1, 2, ..., N-1 is expressible as $x_1 + x_2 + x_3 + x_4$ with each x_i less than N. There are exactly $n_1 \times n_2 \times n_3 \times n_4$ different such expressions, so we must have $n_1 \times n_2 \times n_3 \times n_4 \geq N$.

Thus the geometric mean of $\{n_1, n_2, n_3, n_4\} \geq N^{\frac{1}{4}}$. since the arithmetic mean is never less than the geometric mean for a set of positive real numbers it follows that $\frac{n(N) + 3}{4} \geq N^{\frac{1}{4}}$, or $n(N) \geq 4 N^{\frac{1}{4}} - 3$.

(iii) One choice of X_1, X_2, X_3, X_4 places in X_i all numbers whose decimal representation contains the digit 0 in every place except the "10^{n-1}" places, where $n = 4k + i$, $k = 0, 1, 2, \ldots$.

i.e. $X_1 = \{0, 1, 2, 3, \ldots, 9, 10000, 10001, \ldots 10009, 20000, \ldots\}$

$X_2 = \{0, 10, 20, 30, \ldots, 90, 10000, 10001, \ldots 10009, 20000, \ldots\}$

$X_3 = \{0, 100, \ldots, 900, 1000000, 1000100, \ldots\}$

$X_4 = \{0, 1000, \ldots$

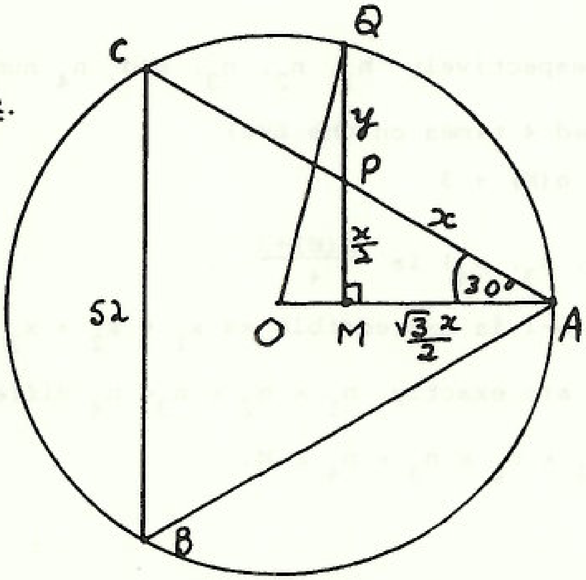
Then, for example, the number m = 5478267316425 is 5000200010005 + 8000300020 + 70007000400 + 400060006000, these summands being in X_1, X_2, X_3, X_4 respectively.

If $N = 10000$, $n(N) = n_1 + n_2 + n_3 + n_4 - 3 = 37 = 4 \times (10000)^{1/4} - 3$.

Q. 715.

In the figure, A B C is an equilateral triangle, with sides 52 cms long, inscribed in a circle. PQ is parallel to BC , and the lengths AP, PQ are x cms, y cms respectively, where x and y are both whole numbers. Find all possible values of x and y.

ANSWER.



From the figure, $OQ^2 = OM^2 + MQ^2$.

$$r^2 = (r - \frac{\sqrt{3}}{2} x)^2 + (\frac{x}{2} + y)^2$$

where $r = \frac{52}{\sqrt{3}}$ is the radius of the circle.

This is equivalent to

$$104^2 = (3x - 104)^2 + 3 (x + 2y)^2$$

$$104^2 = H^2 + 3\ K^2 \qquad (1).$$

$$\text{where} \quad H = 3x - 104 \quad , \quad K = x + 2y.$$

The simplest way to proceed is to calculate $\sqrt{(104^2 - 3K^2)}$ for $K = 1, 2, \ldots, 60$ with a calculator. The only integer solutions prove to be

$$K = 28 \quad , \quad H = \pm 92.$$
$$K = 32 \quad , \quad H = \pm 88$$
$$K = 52 \quad , \quad H = \pm 52$$
$$K = 60 \quad , \quad H = \pm 4.$$

The solutions $H = -92$, $K = 28$ yields $x = 4$, $y = 12$

$H = 4$, $K = 60$ yields $x = 36$, $y = 12$.

Other values of H and K either yield non-integer values of x, y or values of x outside the range $0 < x < 52$.

[One can reduce the amount of trial and error in finding the solutions of (1) in a number of ways. For example, one can establish that all solutions are given by $H = \pm \delta \ \frac{(v^2 - 3u^2)}{2}$, $K = \delta\ u\ v$,

where δ, u, v are whole numbers, such that $\delta\ \frac{(3u^2 + v^2)}{2} = 104$.

Then $(\delta, u, v) = (4, 1, 7) \rightarrow H = \pm 92, K = 28$

$(\delta, u, v) = (16, 2, 1) \rightarrow H = \pm 88, K = 32$

$(\delta, u, v) = (4, 3, 5) \rightarrow H = \pm 4\ , K = 60$

and $(\delta, u, v) = (52, 1, 1) \rightarrow H = \pm 52, K = 52.$]

Q 716. (i) Suppose that x is a real number with the property that there is an infinite sequence of rational numbers.

$$\frac{p_1}{q_1}\ ,\ \frac{p_2}{q_2}\ ,\ \ldots.\ ,\ \frac{p_n}{q_n}\ ,\ \ldots$$

(p_n, q_n are integers), all different from x, but such that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{{q_n}^2} \quad \text{for } n = 1, 2, 3, \ldots.$$

Prove that x cannot be a rational number.

(ii) Let $x = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^7} + \ldots + \frac{1}{2^{2^n - 1}} + \ldots.$

Show that x is an irrational number.

ANSWER. Suppose we are given a real number, x, and we try to approximate it by a rational number with denominator q. The points on the number axis corresponding to $0, \pm 1/q, \pm 2/q, \ldots, \pm h/q, \ldots$ subdivide it into intervals of length $1/q$, and x (if it is not equal to one of them) lies in some interval $h/q < x < (h+1)/q$. Hence there is at most one such rational number (either h/q or $(h+1)/q$), for which $|x - p/q| < 1/2q$.

Since $\frac{1}{q^2} \le \frac{1}{2q}$ when $q > 1$, there is certainly at most one number $\frac{p}{q}$ for which $|x - \frac{p}{q}| < \frac{1}{q^2}$.

Now suppose x itself is a rational number, $x = \frac{h}{k}$, and suppose that there is an infinite sequence, $\frac{p_n}{q_n}$ as in the question.

For each of $q = 2, 3, \ldots, k$ there is at most one number in the sequence having q as denominator. When these are deleted there must remain an infinite sequence in which all the denominators exceed k.
But if $q_n > k$

$$\left|\frac{h}{k} - \frac{p_n}{q_n}\right| = \frac{|h\,q_n - p_n\,k|}{k\,q_n} \ge \frac{1}{kq_n} > \frac{1}{q_n^2}$$

(since the numerator is an integer $\ne 0$). Thus no such sequence can exist if x is rational. [Comment: It is a more interesting fact that for every irrational number x such a sequence of rational numbers can be found.]

(ii) Let $r_n = \frac{1}{2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{2^n-1}}$

$$= \frac{p_n}{2^{2^n-1}} \quad \text{(where } p_n \text{ is some whole number).}$$

Now $x - r_n = \frac{1}{2^{2^{n+1}-1}} + \frac{1}{2^{2^{n+2}-1}} + \ldots$

$$< \frac{1}{2^{2^{n+1}-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right)$$

$$< \frac{1}{2^{2^{n+1}-2}} = \left(\frac{1}{2^{2^n-1}} \right)^2 .$$

Since this is true for all n, it follows from (i) that x must be irrational.

Q 717. (i) The function $f(x) = e^x = \exp(x)$ has the property that $f'(x) = f(x)$ for all x.

Prove that for all positive numbers x

$$e^x > 1 + x.$$

(ii) Let $x_1, x_2, \ldots x_n$ be any positive numbers,

$S_n = x_1 + x_2 \ldots + x_n$, $Q_n = (1+x_1)\times(1+x_2)\times \ldots \times(1+x_n)$.

Show that $\exp(S_n) > Q_n > S_n$

(ii) Let $a_k = \dfrac{k^2 + k + 1}{k^2 + k}$, $p_n = a_1 \times a_2 \times \ldots\ldots \times a_n$.

Show that $p_n < e$ for any positive whole number n.

ANSWER. (i) $e^o = 1$ and $e^x > 0$ for all x $\therefore f'(x) > 0$ for all x, so $f(x)$ is an increasing function. Hence if $x > 0$, $f(x) > e^o = 1$.

$$\text{It follows that } e^x-1 = \int_0^x f'(t)dt = \int_0^x f(t)dt > \int_0^x 1\, dt = x$$

$$\text{i.e. } e^x > 1 + x \text{ for } x > 0.$$

(ii) $\exp S_n = \exp x_1 \exp x_2 \ldots\ldots \exp x_n$

$> (1 + x_1)\ (1 + x_2)\ \ldots\ldots\ (1 + x_n)$ by (i)

$= Q_n$.

and $Q_n = 1 + \sum_{i=1}^{n} x_i + \sum_{1\le i<j\le n} x_i x_j + \ldots\ldots$

$> 1 + S_n$ (since all omitted terms are positve)

$$> S_n .$$

(iii) Take $x_k = \frac{1}{k^2 + k}$ in (ii)

$$Q_n = \prod_{k=1}^{n} \left(1 + \frac{1}{k^2 + k} \right) = \prod_{k=1}^{n} \frac{k^2 + k + 1}{k^2 + k} = P_n .$$

and $$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)}$$

$$= (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{n} - \frac{1}{n+1})$$

$$= 1 - \frac{1}{n+1} < 1.$$

$\therefore$ (ii), $P_n = Q_n < \exp S_n < \exp 1 = e$.

Q. 718. For any positive integer n

$$x_n = (1 + \frac{1}{2}) \times (1 + \frac{1}{3\times 2+3}) \times ..$$

$$\dots \times (1 + \frac{1}{n\times(n-1)\times\dots\times 2 + n\times(n-1)\times\dots\times 3 + \dots + n})$$

Show that $x_n < 2$ for all n.

ANSWER. We shall show by mathematical induction

that $x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$

This is true when $n = 2$ since the only factor is $1 + \frac{1}{2}$.

Assuming $x_{k-1} = 1 + \frac{1}{2!} + \dots + \frac{1}{(k-1)!}$

$$x_k = x_{k-1} \times \left(1 + \frac{1}{k\times(k-1)\times - \times 2 + k \times \dots \times 3 + \dots + k} \right)$$

$$= x_{k-1} \times \left(1 + \frac{1}{k! \; (\frac{1}{1} + \frac{1}{2!} + \dots \frac{1}{(k-1)!}} \right)$$

$$= x_{k-1} + \frac{1}{k!} = 1 + \frac{1}{2!} \dots + \frac{1}{(k-1)!} + \frac{1}{k!}$$

The induction step has been established. Hence

$$x_n = 1 + \frac{1}{2!} + \dots + \frac{1}{n!} \quad \text{for all } n \ge 2$$

Since $\frac{1}{k!} \le \frac{1}{k^2}$ for $k \ge 2$,

$$x_n \le 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = 2\left(1 - \frac{1}{2^{n+1}}\right) < 2.$$

__Q. 719__. For any integer n each of $f(n)$, $g(n)$, and $h(n)$ denotes a corresponding integer. No two of f, g, and h are identical functions. (e.g. There is at least one integer m such that $f(m) \neq g(m)$). Each of f, g, and h satisfies the functional equation:-

$$F(mn) = F(m)\,F(n) + F(m+n) - 1 \quad \text{for all integers } m,\ n.$$

(e.g., $h(mn) = h(m)\,h(n) + h(m+n) - 1$ for all integers m, n).
If $f(1987) = f(1)$ and $g(1987) = g(0)$, find $h(1987)$.

__ANSWER__. We try to find all possible solutions, $F(n)$, of

$$F(mn) = F(m)\,F(n) + F(m+n) - 1 \quad (1).$$

Setting $m = n = 0$ in (1), $F(0)^2 = 1$ $\therefore F(0) = \pm 1$.
If $F(0) = 1$, taking $m = 0$ in (1)

$$1 = 1.\ F(n) + F(n) - 1$$

$$\therefore F(n) = 1 \text{ for all } n.$$

This gives one solution of the functional equation, (1).

For all other solutions $F(0) = -1$

Set $m = n = 2$ in (1). $F(2)^2 = 1$

$$\therefore F(2) = \pm 1.$$

__Case A__ $F(0) = F(2) = -1$.

Set $m = n = 1$ in (1). $F(1) = (F(1))^2 - 2$

$$\therefore F(1) = -1 \text{ or } F(1) = 2$$

If $F(1) = -1$, setting $m = 1$ in (1) yields

$$F(n) = -F(n) + F(n+1) - 1$$

Hence if either $F(n)$ or $F(n+1) = -1$, so is the other. Hence $F(n) = -1$ for all n is a second solution of (1).

If $F(1) = 2$, setting $m = 1$ in (1) yields

$$F(n) = 2\,F(n) + F(n+1) - 1$$

$$F(n+1) = 1 - F(n) \quad \text{for all } n.$$

If either of $F(n)$, $F(n+1)$ is -1 or 2 the other takes the remaining value. This yields the solution

$$F(n) = -1 \quad \text{for } n \text{ even}$$

$$F(n) = -2 \quad \text{for } n \text{ odd}$$

Case B $F(0) = -1$; $F(2) = 1$

Set $m = n = 1$ in (1). $F(1) = F(1)^2$

$$\therefore \; F(1) = 1 \text{ or } 0.$$

If $F(1) = 1$, setting $m = 1$ in (1) gives $F(n) = 1$ for all n.

(This is inconsistent with $F(0) = -1$).

$\therefore$ if $F(0) = -1$, and $F(2) = 1$ we must have $F(1) = 0$ and setting $m = 1$ in (1) yields $F(n+1) = F(n) + 1$.

Since $F(1) = 0$ we obtain $F(n) = n - 1$ for all n.

We have shown that if $F(n)$ satisfies (1) there are four possibilities for $F(n)$:-

(A) $F(n) = 1$ for all n ;(B) $F(n) = -1$ (all n);

(C) $F(n) = -1$ (n even), $F(n) = 2$ (n odd);

(D) $F(n) = n-1$, all n.

(It can be immediately verified that each of these actually does satisfy (1) for all m, n).

Now $g(n)$ can be either of (A) or (B), and $f(n)$ can be any one of (A), (B) or (C). So we are unfortunately unable to decide which function is $h(n)$; it could be any of the above four solutions. There are thus 4 possible answers for $h(1987)$; viz 1, -1, 2, 1986.

[Comment. When I constructed the problem, I missed one of the solutions of (1), and believed $h(1987)$ was uniquely determined. To amend the question, add an extra function $k(n)$, such that $k(1987) = k(2)$. Now none of f,g, and k is the solution (D) above, so we must have $h(1987) = 1987 - 1 = 1986$. Apologies, Ed.].