Parabola Volume 24, Issue 1 (1988)

THE UNIT FRACTION

Paul Erdos*

For reasons which are not entirely clear to me the ancient Egyptians considered fractions $\frac{1}{b}$ much simpler than fractions of the form $\frac{a}{b}$ and they were interested in representing

(1) $$\frac{a}{b} = \frac{1}{x_1} + --- + \frac{1}{x_n}$$

where a and b are positive integers and $x_1 < x_2 .. < x_n$ is an increasing sequence of integers. The first proof that (1) is always solvable is due to Fibonacci from the 13th century. He is also sometimes known under the name Leonardo de Pisa (not to be confused with Leonardo da Vinci (1492-1519)).

Fibonacci proved (1) by the so called "greedy algorithm". We always subtract from $\frac{a_i}{b_i}$ the largest fraction $\frac{1}{x_i}$ for which

$$\frac{a_i}{b_i} - \frac{1}{x_i} > 0 .$$

A simple calculation (think how x_i is defined in terms of $\frac{bi}{a_i}$) gives that

$$\frac{a_i}{b_i} - \frac{1}{x_i} = \frac{a_{i+1}}{b_{i+1}} \text{ where } 0 \le a_{i+1} < a_i .$$

*George Szekeres writes abou Paul Erdos at the end of this article.

Since the a_i's form a strictly decreasing sequence of non-negative integers this reduction must eventually come to an end, namely when $a_{i+1} = 0$. Thus by using this greedy algorithm we immediately obtain that

$$\frac{a}{b} = \frac{1}{x_1} + \dots + \frac{1}{x_n}, \quad x_1 < x_2 < \dots < x_n$$

is always solvable for $n \le a$. The smallest n for which (1) is solvable is a difficult problem. It is well known and easy to prove that for a = 3 there are infinitely many integers b for which

$$\frac{3}{b} = \frac{1}{x_1} + \frac{1}{x_2}$$

has no solutions. On the other hand 40 years ago Straus and I conjectured that for every n > 9

$$\frac{4}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \quad 1 < x_1 < x_2 < x_3$$

is always solvable. This question is probably very difficult and I offer a prize of 250 dollars for the first proof or counterexample. (See Editorial.)

An old theorem (I do not know who first proved it, the interested reader can no doubt find it in the monumental work of L. Dickson, "History of the theory of numbers") states that

$$(2) \qquad \frac{1}{a+1} + \frac{1}{a+2} + \dots + \frac{1}{b} = \sum_{a<t\le b} \frac{1}{t} = n$$

can never hold. The original proof of (2) goes as follows: first of all observe that if (2) holds we must have b > 2a. To see this observe that since n is an integer we must have

$$\frac{1}{a+1} + \dots + \frac{1}{b} \ge 1 \text{ and } \frac{1}{a+1} + \dots + \frac{1}{2a} < \frac{a}{a+1} < 1$$

Thus $b > 2a$ is proved. Now a well known theorem of Tchebicheff asserts that for every a there is always a prime p between a and $2a$. Let now p be a prime between $\frac{b}{2}$ and b, i.e., $\frac{b}{2} \leq p<b$. Clearly from $b > 2a$ we have $p > a$. Given $2p > b$ in the sum $\sum_{a<t\leq b} \frac{1}{t}$ there is only one integer t which is a multiple of p. Now put

$\frac{1}{a+1} + \ldots + \frac{1}{b} = \frac{1}{p} + \frac{u}{v}$, where v is not divisible by p, v is the least common multiple of the integers $a < t \leq b$, $t \neq p$, thus v is not a multiple of p. Now

$$\frac{1}{a+1} + \ldots + \frac{1}{b} = \frac{1}{p} + \frac{u}{v} = \frac{v + p\,u}{p\,v}$$

Clearly the numerator $v + pu$ is not a multiple of p thus $\frac{v + p\,u}{pv}$ can not be an integer. Thus

$$\Sigma_{a,b} = \frac{1}{a+1} + \text{--} + \frac{1}{b}$$

can not be an integer and our theorem is proved. This proof is very simple but it uses the theorem of Tchebicheff. Kurschak noticed more than 60 years ago that a still simpler proof is possible. To see this let 2^n be the unique power of 2 in $\frac{b}{2} < 2^n \leq b$. Clearly no integer other than 2^n is a multiple of 2^n in (a,b). Now write

$$\Sigma_{a,b} = \frac{1}{2^n} + \left(\Sigma_{a,b} - \frac{1}{2^n} \right) = \frac{1}{2^n} + \frac{u}{v} \tag{3}$$

where v is the least common multiple of the integers $t \neq 2^n$, $a<t\leq b$. Thus v is not a multiple of 2^n. Now from (3) we have

$$\Sigma_{a,b} = \frac{v + 2^n\,u}{2^n\,v}$$

Clearly $\Sigma_{a,b}$ can not be an integer since $v + 2^n u$ is not a multiple of 2^n.

I hope you agree with me that this proof is as simple as possible and is also very beautiful. Now I would like to state two old conjectures of mine: Assume

$$(4) \qquad \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = 1, \quad 1 < x_1 < x_2 < \quad < x_n$$

then by what we just proved

$$\max (x_m - x_i) \geq 2.$$

I conjecture that if (4) holds then in fact

$$(5) \qquad \max (x_{i+1} - x_i) \geq 3$$

(5) if true is best possible since

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1 .$$

Also it can be shown by elementary analysis that

$$\Sigma_{a,b} = \log \frac{b}{a} + \varepsilon$$

where ε is arbitrarily small for large a.

Thus if (4) holds then $\frac{x_n}{x_1} \geq e$ where e is the basis of the natural logarithm.

Perhaps for every $\varepsilon > 0$ there are integers a and b, $b < (a + \varepsilon)e$ for which

$$\Sigma_{a,b} = 1 .$$

A rich source of further problems on Egyptian (or unit) fractions is the little book by R.L. Graham," Old and new problems and results in combinatorial number theory".

---

*Hungarian-born Paul Erdos is the most prolific mathematician of our times, at latest count well over 1000 mathematical papers to his credit, a record unsurpassed by anyone in modern times (with the possible exception of Leonhard Euler in the 18th century). At 75, he travels tirelessly all over the world (including Australia), inundating his countless collaborators with a seemingly inexhaustible supply of new and interesting problems, many of them created at the spur of the moment, while taking a walk or sitting at the dinner table with friends and colleagues.

There are innumerable stories going around in the mathematical world about Erdos, not in the least because of his uninhibited behaviour and unconventional ways in which he expresses himself in lectures and everyday conversation. His friends created the concept of Erdos number: you have Erdos number 1 if you have a joint article with him, 2 if you don't have such an article but have a joint paper with someone who has Erdos number 1, so on. Your Erdos number is the smallest number of steps you can reach him through a chain of joint articles. In this fashion you get a connected graph whose properties have not been fully investigated but probably would include a large percentage of mathematicians (dead or alive). This writer boasts to have the earliest Erdos number 1 (certainly the oldest among living mathematicians) dating back to 1934. The present article by Erdos is a contribution to Parabola on his most recent visit to Sydney.

Another Problem of Ages

One morning after church the verger, pointing to three departing parishioners, asked the bishop, "How old are those three people?"

The bishop replied, "the product of their ages is 2,450, and the sum of their ages is twice your age."

The verger thought for some moments and said, 'I'm afraid 1 still don't know."

The bishop answered, "I am older than any of them."
"Aha!" said the verger, "Now I know."

<u>Problem</u>: How old was the bishop? (Ages are in whole numbers of years and no one is over 100).