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SPACE, GEOMETRY, CURVATURE

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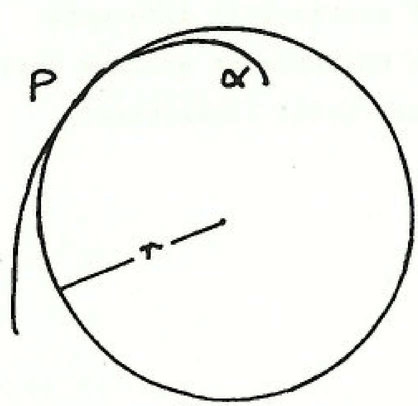
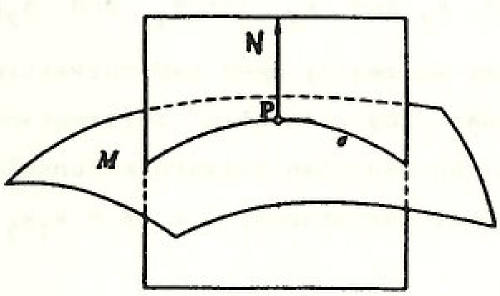
If one wishes to appear wise in the eyes of one's friends (and at times this may not be disadvantageous) it is generally sufficient to inform them (in an appropriate casual way) that space is not Euclidean, that in fact it is "curved". Although it is a little difficult to explain what one means by this profundity it is not particularly difficult to at least give a few clues. One can do this to some extent by explaining how curvature is defined for curves and surfaces and "showing" what properties it has in this context. One can then give some further explanation of why the concept is so important in Einstein's world.

Let's begin by talking about the curvature of a plane curve. Apart from the straight line the simplest curve is a circle. What should we define the curvature of a circle to be? We agree that the bigger the radius of the circle is the flatter the circle is, and therefore the smaller its curvature should be. We are arguing that the circle's curvature, κ, should have an inverse relationship with the radius. It thus natural to define $\kappa = 1/r$ where r = radius of the circle. Notice that if we consider a straight line to be the limiting case of a circle of arbitrarily large radius we have the result that a straight line has zero curvature. Now suppose we wish to talk about the curvature of an arbitrary curve α at a point P. Let's suppose we can approximate α at P by a circle. Given what we have just decided it is reasonable to define the curvature, $\kappa_\alpha(P)$, of α at P to be $1/r$ where r is the radius of the approximating circle. Of course as P varies so does r so in fact κ_α is generally not a constant function.



There remains the minor technical hitch of explaining what we mean when we say "approximate α at P by a circle." Imagine the curve α as a road and imagine that we are driving along it at some constant speed v. When we go round the curve at P the car experiences a sideways, or normal, force (which hopefully is equalled by the frictional force on the tyres). The approximating circle has the property that the car would experience exactly the same normal force if it were travelling around that circle at the same speed.

It is decidedly more non-trivial to decide how to define the curvature of a surface M at a point P. Without worrying about the mathematical technicalities of the definition of a surface hopefully we can recognize a surface when we see one. Soap bubbles are examples of surfaces par excellence. Boundaries of solids also generally form surfaces. For example consider the surface of a ball bearing, the surface of a coffee jar or the surface of a doughnut. The critical idea mathematically is that a surface should look smooth and flat at least "locally" - after all the flat earthers are nearly right. Anyway let us denote our favourite surface by M with P an arbitrary point on it. (Why not imagine the surface of a rugby ball with P the top of the ball - if you have an aversion to rugby call the ball an ellipsoid.)



Any plane normal (perpendicular) to M at P will cut M near P along a curve σ called a <u>normal section</u>. As we turn such a plane still making sure it contains the normal N to M at P we get a family of normal sections each of which has its own curvature at P. It is not difficult to show (this was known by Euler circa 1780) that at such a point P there exist two distinguished normal sections σ_2, σ_2 which are such that

(1) they are mutually perpendicular;

(2) the curvatures k_1 and k_2 of σ_1 and σ_2 are the minimum and maximum values of the curvatures of all normal sections. (We allow k_1 and k_2 to have opposite signs if the curves are bending in opposite directions as at a

saddle point. In the particular case $k_1 = k_2$ the curvature of all normal sections are the same, as is the case, for example, on a sphere.)

The directions of σ_1 and σ_2 are called the <u>principal directions</u> and the curvatures k_1 and k_2 are called the <u>principal curvatures</u> of the surface at P.

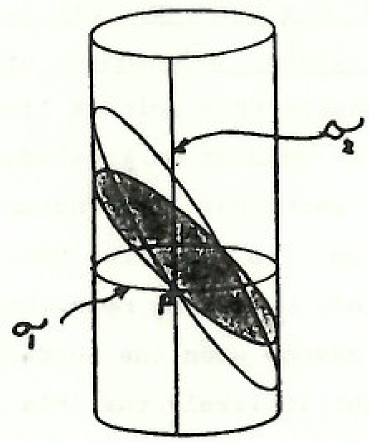
If we consider the case of the football it is clear that one principal direction is along the top of the ball with the stitching whilst the other is at right angles. If the surface were a saddle we can see that one normal section would be defined by the stirrups whilst the other would be at right angles in the direction of the horses back and neck.

The surprising thing is that this phenomenon is not a property of the any symmetry of the surface. If one considers the skin of one's body then no matter what point we consider on our body it is easy to spot the directions of extreme curvature; they will always be at right angles.

Euler in fact was able to show that the curvature of any planar curve in M through P could be described in terms of k_1 and k_2 (or σ_1 and σ_2). Consequently one would be inclined to think that we really need two curvatures functions and not be inclined to throw away any of this information. Nevertheless we gain much if we concentrate on the Gaussian curvature function K. It is defined to be the product of the principal curvatures, i.e. $K = k_1 k_2$.

For a plane $k_1 = k_2 = 0$ so $K = 0$ whilst for a sphere of radius r we have $k_1 = k_2 = \frac{1}{r}$ (every normal section is a great circle of radius r) so $K = \frac{1}{r^2}$.

For a circular cylinder of radius r one normal section is a circle of radius r (so $k_1 = \frac{1}{r}$ say) whilst the other is a straight line parallel to the axis of the cylinder (so $k_2 = 0$). Consequently $K = 0$ for the cylinder.



Why then is Gaussian curvature so important? The above examples of the plane and cylinder give a clue. If we have a flat piece of paper we can wrap it into a cylinder - in an essential way they are the same surface. Perhaps K doesn't change if we deform a surface in such a way that it is neither stretched nor torn. Indeed this is the case and the result is generally known as the Theorema Egregium - it was a favourite of Gauss and he proved it about 1820.

Theorema Egregium Gaussian curvature is a bending invariant.

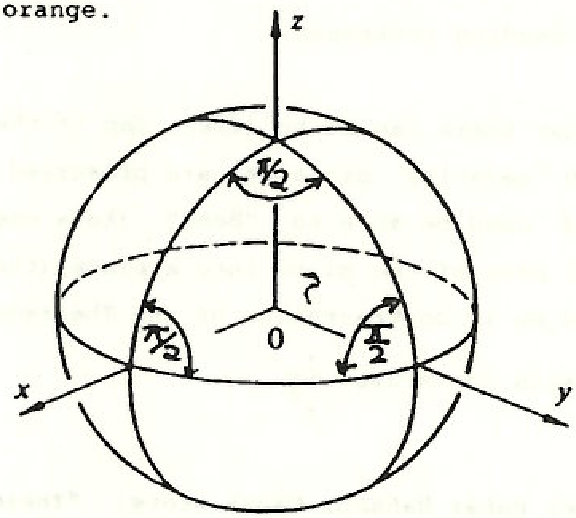
We immediately make the observation that there can be no (flat) map of the globe (or part of the globe) in which relative distances are preserved. Afterall if we could do this we would need be able to "bend" (here one shouldn't think too much in physical terms) part of the globe into a plane (the scaling is not important). But this would be in contravention of the Theorema Egregium since the sphere has non-zero Gaussian curvature $\frac{1}{r^2}$.

In 1825, in a letter to the astronomer Peter Hansen, Gauss wrote: "These investigations deeply affect many other things; I would go so far as to say they are involved in the metaphysics of the geometry of space."

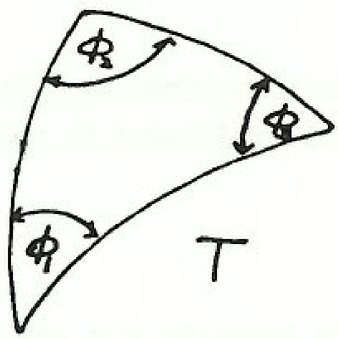
So here we have Gauss - to many the greatest mathematician of all time - anticipating the success of Einstein by some 80 years. Of course Gauss didn't have any divine knowledge of what Einstein was going to achieve. However Gauss was different from his contemporaries in at least two ways. He was convinced that the laws of geometry (as applying to the real world about us) had to be discovered emperically. Further he realized from his own work that it was possible to study surfaces as entities in their own right, independent of how they may be embedded in Euclidean 3-space. To clarify this point a little, observe that our definition of Gaussian curvature K depends on knowledge of the normal N to the surface (to obtain the normal sections). The normal N points out of the surface into the surrounding space. To prove the Theorema Egregium Gauss showed that one could compute K solely by taking measurements inside the surface (since these measurements do not change when the surface is bent K will not change). Moreover he clearly thought it likely that his work could be generalized to higher dimensions. If fact his successor at Gottingen, Bernhard Riemann did just this. Consequently much of the mathematical framework for Einstein's general theory had been developed by the mid nineteenth century.

If it is a little beyond us to consider a 4-dimensional manifold with metric

$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$ we can still learn much by contemplating an orange.



On a sphere there are of course no straight lines. On a surface the curves that play the role of straight lines are the curves of shortest length joining 2 points. These lines are called geodesics. On a sphere it is "clear" that the geodesics are segments of great circles, i.e. circles whose centres are the centre of the sphere. With these preliminiary observations out of the way let's consider a geodesic triangle which makes up one eight of the sphere. The sum of its interior angles is not π but $3\pi/2$. This excess in angle is presumeably explained in terms of the curvature of the triangle. What is the total curvature of the triangle? It's area is $\frac{1}{8} \times 4\pi r^2$ and its (Gaussian) curvature is $\frac{1}{r^2}$, so that its total curvature is $\frac{1}{8} \times 4\pi r^2 \times \frac{1}{r^2} = \pi/2$. At least for the geodesic triangle T we are considering



the sum of the internal angles $\phi_1 + \phi_2 + \phi_3$ $= \pi$ + total curvature of T. Indeed this is a general theorem for geodesic triangles. If the triangle doesn't have geodesics for sides the result generalizes to:

$\phi_1 + \phi_2 + \phi_3 = \pi +$ <u>total curvature of</u> T + <u>total geodesic curvature of T's sides</u>.

We will not explain what we mean by "geodesic curvature" except to say it's the curvature of a curve as an inhabitant of the surface itself would measure it.

We finish by generalizing from a simple observation. We observe that the total (Gaussian) curvature of a spherical balloon of radius r is $4\pi r^2 . \frac{1}{r^2} = 4\pi$. The key observation is that the answer does not depend on r. Is this a fluke? Suppose we were to squeeze it. Would the total curvature change? In fact the answer is no. Again this is a specific case of a result which is a little difficult to state in full generality. It implies, for example, that all inner tubes have total curvature 0 no matter who is sitting on them. In any case such a result is saying that the total curvature of a closed surface is invariant under very general deformations. What are the common invariances of nature? We certainly know that energy is conserved. Is there a connection between these two invariances? The answer is yes. In the real world, in Einstein's world, curvature is due to mass. Our results about the conservation of the total curvature of a balloon or of an inner tube in point of fact correspond to the conservation of mass (or energy) in general relativity.

E.T. Bell's "Men of mathematics" provides a romantic account of these mathematicians. For an extended account of space and geometry "Space through the ages" by Cornelius Lanczos is a terrific read.