PROBLEM SECTION

Students are invited to submit solutions of any of these problems to PARABOLA. Correct solutions and the names of the solvers will be published in the next issue.

Q.732. Let L be a set of n line segments with the property that any three of them can be assembled to form a triangle. A pair of line segments is called "exceptional" if one is more than twice as long as the other. What is the maximum possible number of exceptional pairs in L?

Q.733. Let three lengths $a,b,c,$ and a point P be given. It is desired to construct an equilateral triangle ABC with P as an interior poiont such that the line segments PA, PB, PC are respectively a, b and c.

(i) Find conditions on a, b, c, for the construction to be possible.

(ii) Show how the construction can be performed with straight edge and compass.

Q.734. (i) Let S be a set of rational numbers with the property that the product of every two distinct elements of S is an integer. Show that the product of every k distinct elements of S is an integer for all $k > 2$.

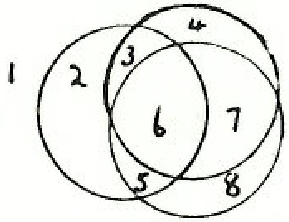
(ii) Show that (i) becomes false if the word "rational is omitted.

Q.735. Prove that the equation

$$x^{1988} - 2x^{1987} + 3x^{1986} + \dots + 1987x^2 - 1988x + 1989 = 0$$

has no real root.

Q.736. One circle divides the plane into 2 regions; two distinct circles give three or 4 regions, depending on their relative position. Three circles can yield 8 regions, but not more.



Find a formula for the maximum number of regions obtainable from n circles, and prove your result.

Q.737. Find all real numbers x such that

$$\sqrt{x + 2 - 4\sqrt{x-2}} + \sqrt{4x - 7 - 4\sqrt{x-2}} = 2 + \sqrt{x - 1 - 2\sqrt{x-2}}$$

(As usual, $\sqrt{y}$ denotes the non-negative square root of y)

Q.738. ABCDE is a tetrahedron having opposite sides of equal length (i.e. AB = CD, AC = BD, AD = BC). Prove that the faces of the tetrahedron are acute angled triangles.

Q.739. A random number generator churns out the sequence $x_1, x_2, \ldots, x_n, \ldots$ where each x_i is one of 1, 2, 3, ..., 9 all with equal probability. Let y_n be the product $x_1 x_2 \ldots x_n$. Find the probability that y_n is divisible by 10.

Q.740. A deck contains N cards, of which 3 are kings. This deck is shuffled thoroughly (i.e. until all possible arrangements are equally likely) and then the cards are turned up one by one from the top until the second king appears. If this procedure is repeated many times prove that the average number of cards turned up is likely to be close to $\frac{(n + 1)}{2}$.

Q.741. v, w, x, y, z are real numbers such that

$$v + w + x + y + z = 11$$

and $v^2 + w^2 + x^2 + y^2 + z^2 = 25$

Find the largest possible value of z.

Q.742. If x is a real number, denote by [x] the integral part of x (that is, the largest integer not greater than x). Find a positive integer n

(i) such that

$$[1^{1/3}] + [2^{1/3}] + [3^{1/3}] + \dots + [n^{1/3}] = 500$$

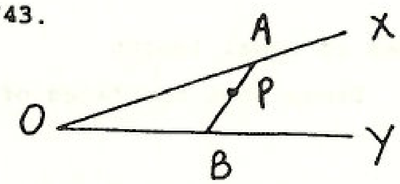
(ii) Show that

$$[1^{1/2}] + [2^{1/2}] + [3^{1/2}] + \dots\dots [(m^2 - 1)^{1/2}]$$

$$= \frac{1}{6} m(m-1) (4m+1)$$

for all positive integers m.

Q.743.



Given an angle $X\hat{0}Y$ and a point P within its arms, show how to construct points A,B on the arms such that APB is straight and the triangle ΔAOB is of minimum area.

SOLUTIONS OF PROBLEMS Q.720 - Q.731

Q.720. When the initial digit of a whole number x is deleted, the number decreases by a factor of 13.
Find all possible values of x.

ANSWER. Let $x = z \times 10^n + y$ where z is the initial digit of x, and $0 \le y < 10^n$. We wish to find y and z such that

$$z \times 10^n + y = 13y$$

i.e.

$$12y = z \times 10^n.$$

Since 3 is a factor of the L.H.S., but not of 10^n we must have $z = 3$, or 6, or 9.

If $z = 3$, $y = 25 \times 10^{n-2}$ and $x = 325 \times 10^{n-2}$, $n \ge 2$.

If $z = 6$, $y = 5 \times 10^{n-1}$ and $x = 65 \times 10^{n-1}$, $n \ge 1$.

If $z = 9$, $y = 75 \times 10^{n-2}$ and $x = 975 \times 10^{n-2}$, $n \ge 2$.

Hence the possible values of x are 65, 325, 975 or the numbers obtained by following these with any number of zeros.