$$[1^{1/3}] + [2^{1/3}] + [3^{1/3}] + \dots + [n^{1/3}] = 500$$

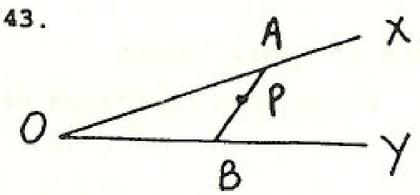
(ii) Show that

$$[1^{1/2}] + [2^{1/2}] + [3^{1/2}] + \dots\dots [(m^2 - 1)^{1/2}]$$

$$= \frac{1}{6} m(m-1)\ (4m+1)$$

for all positive integers m.

Q.743.



Given an angle $X\hat{O}Y$ and a point P within its arms, show how to construct points A,B on the arms such that APB is straight and the triangle ΔAOB is of minimum area.

SOLUTIONS OF PROBLEMS Q.720 - Q.731

Q.720. When the initial digit of a whole number x is deleted, the number decreases by a factor of 13.

Find all possible values of x.

ANSWER. Let $x = z \times 10^n + y$ where z is the initial digit of x, and $0 \le y < 10^n$. We wish to find y and z such that

$$z \times 10^n + y = 13y$$

i.e.

$$12y = z \times 10^n.$$

Since 3 is a factor of the L.H.S., but not of 10^n we must have z = 3, or 6, or 9.

If $z = 3$, $y = 25 \times 10^{n-2}$ and $x = 325 \times 10^{n-2}$, $n \ge 2$.

If $z = 6$, $y = 5 \times 10^{n-1}$ and $x = 65 \times 10^{n-1}$, $n \ge 1$.

If $z = 9$, $y = 75 \times 10^{n-2}$ and $x = 975 \times 10^{n-2}$, $n \ge 2$.

Hence the possible values of x are 65, 325, 975 or the numbers obtained by following these with any number of zeros.

Correct solution from F. Antonuccio, St Gregory's College, Campbelltown.

Q.721. When the initial digit of a whole number x is deleted, the number decreases by a factor of 13.
Find all possible values of x.

ANSWER. Again let $x = z \times 10^n + y$ where z is the initial digit of x, and $0 \le y < 10^n$. We wish to find k such that

$$z \times 10^n + y = ky \quad \text{has solutions } z, y.$$

i.e.
$$(k-1)y = z \times 10^n.$$

Clearly $k-1 > z$, since $10^n > y$. (*)
Since any prime factor of (k-1) other than 2 or 5 is not a factor of 10^n, it must be a factor of the digit z. Hence the only possible values of (k-1) are $2^a 5^b$, $3.2^a.5^b$, $7.2^a.5^b$, and $9.2^a.5^b$ where a and b are non-negative integers, not both zero because of (*).
For each of these possible values of $k = q \times 2^a \times 5^b + 1$ one must choose any digit z which is a multiple of q and then obtain

$$x = z \frac{k}{k-1} 10^n \text{ where } n \ge \max\{a,b\}.$$

Correct solution from F. Antonuccio, who also showed that if $k \le 10z + 1$, there is a solution x whose second digit is not 0.

Q.722. A list of numbers $\{x_1, x_2, x_3, x_4, \ldots, x_n, \ldots\}$ is constructed as follows:-
Any four positive whole numbers less than 100 are chosen for x_1, x_2, x_3 and x_4. For $n > 4$, x_n is the number formed from the last two digits of the sum of the previous 4 numbers. e.g. the list starting {21, 73, 86, 20, ... would continue ..., 0, 79, 85, 84, 48, ...}. Is it possible that the number x_1 never occurs a second time in the list? Prove your assertion.

Answer. We shall prove that x_1 is certain to occur infinitely often in the list.

There are only 10^8 different eight digit numbers, and therefore 10^8 different possible ordered sets of four 2-digit numbers. Hence, when $10^8 + 4$ terms of the list have been constructed, it is impossible that all sets of 4 consecutive numbers

$$\ldots x_j \; x_{j+1} \; x_{j+2} \; x_{j+3} \ldots$$

$j = 1, 2, \ldots, 10^8 + 1$ are different.

Suppose that $\ldots x_s, x_{s+1}, x_{s+2}, x_{s+3} \ldots$ and $\ldots x_t, x_{t+1}, x_{t+3}, \ldots$ are identical such sets (where $t > s$). We see that we must haveA $\underset{s-1}{!A_1!0}$ x_{t-1} since there cannot be more than one 2-digit whole number, y, such that

$$y + x_s + x_{s+1} + x_{s+2} - x_{s+3}$$

is a multiple of 100 (possibly 0).

Similarly we can now show that $x_{s-2} = x_{t-2}$ and so on, until we obtain $x_1 = x_{t-s+1}$.

In fact our working shows that the list cycles, that $t-s$ is a period, and hence that x_1 occurs repeatedly.

Q.723. A chain has N links. Seven appropriately chosen links are cut enabling the chain to be separated into pieces. If x is any whole number not exceeding N, it is possible to find some of the pieces containing altogether exactly x links.

Find the largest possible value of N.

Answer. There are seven single separated links (viz the ones which have been cut) and their removal has resulted in another eight lengths of chain, $L_1, L_2, \ldots L_8$, containing respectively $n_1, n_2, \ldots, n_8$ links, where we have labelled the pieces so that $n_1 \le n_2 \le \ldots \le n_8$.

Obviously we can use the cut links to obtain any number up to 7. We will not be able to obtain 8 links unless $n_1 \le 8$. Having chosen the value of n_1, we can now obtain any number of links up to $n_1 + 7$ by using L_1 and the cut links. We will not be able to obtain $n_1 + 8$ unless $n_2 \le n_1 + 8$. Provided this is satisfied we will then be able to obtain any number of links up to $n_1 + n_2 + 7$ using L_1, L_2 and the cut links.

Proceeding in this way we see that we will be able to obtain any number of links up to $N = n_1 + n_2 + \ldots + n_8 + 7$ provided

$n_i \le n_1 + n_2 + \quad + n_{i-1} + 8$ for $i = 2, 3, \ldots 8$.

The largest possible value of N obtainable clearly results from choosing each n_i as large as possible; viz $n_1 = 8$ and

$$n_i = n_1 + n_2 + \ldots + n_{i-1} + 8 \text{ for } i = 2,3, \ldots 8.$$

We calculate $n_2 = 8 + 8 = 16$, $n_3 = 8 + 16 + 8 = 32$, $n_4 = 64$, ...,

$n_k = 2^{k+2}$, ... and $N = 7 + 8 + 16 + \ldots \quad + 2^{10} = 2047$.

Comment: If a cut link can be disengaged from only one of its neighbours (e.g. if each link is shaped like an 8), the above working is not valid. Then one obtains

$$N = 1 + 2 + 4 + \ldots + 2^7 = 255.$$

This view of the problem was taken by F. Antonuccio.

Q.724. Let P(x) denote the polynomial

$$\binom{2k+1}{1}(1 - x^2)^k x - \binom{2k+1}{3}(1 - x^2)^{k-1}x^3$$
$$+ \binom{2k+1}{5}^{k-2}(1 - x^2)^{k-2}x^5 \ldots . + (-1)^k\binom{2k+1}{2k+1}x^{2k+1}$$

(k denotes any positive integer. The notation $\binom{n}{r}$ denotes the binomial coefficient for which ${}^nC_r$ is also sometimes used.)

Use de Moivre's theorem to show that

$$P(\sin\alpha) = \sin(2k+1)\alpha.$$

Deduce that $P(x)$ factorizes as follows:-

$$P(x) = (-1)^k 2^{2k} x\left(x^2 - \sin^2\frac{\pi}{2k+1}\right)\left(x^2 - \sin^2\frac{2\pi}{2k+1}\right)\left(x^2 - \sin^2\frac{3\pi}{2k+1}\right)\cdots$$

$$\cdots\left(x^2 - \sin^2\frac{k\pi}{2k+1}\right).$$

ANSWER. Replacing x by $\sin\alpha$, $1 - x^2$ by $\cos^2\alpha$

$$P(\sin\alpha) = \binom{2k+1}{1}\cos^{2k}\alpha\sin\alpha - \binom{2k+1}{3}\cos^{2k-2}\alpha\sin^3\alpha +$$

$$\binom{2k+1}{5}\cos^{2k-4}\alpha\sin^5\alpha \;.....\; + (-1)^k\binom{2k+1}{2k+1}\sin^{2k+1}\alpha \qquad (1)$$

We are asked to use de Moivre's Theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, to prove that this expression is $\sin(2k+1)\alpha$.

Take $n = 2k+1$ in the theorem and observe that $\sin(2k+1)\alpha$ is the imaginary part of $(\cos\alpha + i\sin\alpha)^{2k+1}$.

By the Binomial Theorem

$$(\cos\alpha + i\sin\alpha)^{2k+1} = \cos^{2k+1}\alpha + \binom{2k+1}{1}\cos^{2k}\alpha(i\sin\alpha)$$

$$+\binom{2k+1}{2}\cos^{2k-1}\alpha(i\sin\alpha)^2 + \;.....\; + \binom{2k+1}{r}\cos^{2k+1-r}\alpha(i\sin\alpha)^r +$$

$$.....\; + \binom{2k+1}{2k+1}(i\sin\alpha)^{2k+1}.$$

The rth term on the R.H.S. is real if r is even, and imaginary if r is odd. It is clear that the terms with r odd yield i times the R.H.S. of (1), since $i^3 = -i$, $i^5 = i$, $i^7 = -i$, ... etc.

Hence $P(\sin\alpha) = \sin(2k+1)\alpha$, as required.

It follows that $x = \sin\alpha$ is a zero (or root) of $P(x)$ if $\sin(2k+1)\alpha = 0$. In particular, this is the case when $\alpha = 0, \pm\frac{\pi}{2k+1}$, $\pm\frac{2\pi}{2k+1}, \ldots, \pm\frac{k\pi}{2k+1}$. Thus $x = \sin 0 = 0, \pm\sin\frac{\pi}{2k+1}, \ldots,$ $\pm\sin$ are $(2k+1)$ roots of $P(x)$, all different. Since $P(x)$ has degree $2k+1$, these are <u>all</u> the roots. By the Factor theorem $\left(x - \sin\frac{r\pi}{2k+1}\right)$ is a factor of $P(x)$ for $r = 0, \pm 1, \ldots \pm k$.

Since $\left(x - \sin\frac{r\pi}{2k+1}\right)\left(x - \sin\frac{-r\pi}{2k+1}\right) = \left(x^2 - \sin^2\frac{r\pi}{2k+1}\right)$ we now have

$$P(x) = Kx\left(x^2 - \sin^2 \frac{\pi}{2k+1}\right)\left(x^2 - \sin^2 \frac{2\pi}{2k+1}\right) \ldots \left(x^2 - \sin^2 \frac{k\pi}{2k+1}\right)$$

where K is a numerical constant yet to be determined. Clearly K is the coefficient of x^{2k+1} in P(x)

$$K = (-1)^k\left[\binom{2k+1}{1} + \binom{2k+1}{3} + \binom{2k+1}{5} + \ldots + \binom{2k+1}{2k+1}\right]$$

$$= (-1)^k \times \frac{1}{2}\left[\binom{2k+1}{0} + \binom{2k+1}{1} + \ldots + \binom{2k+1}{2k+1}\right] \left(\text{since} \binom{2k+1}{r} = \binom{2k+1}{2k+1-r}\right)$$

$$= (-1)^k \times \frac{1}{2} \times (1+1)^{2k+1} = (-1)^k\, 2^{2k}.$$

Correct solution from F. Antonuccio.

Q.725. Assuming the result asserted in Q.724 show that for any positive integer k

(i) $\sin \frac{\pi}{2k+1} \cdot \sin \frac{2\pi}{2k+1} \cdot \sin \frac{3\pi}{2k+1} \ldots \sin \frac{k\pi}{2k+1} = \frac{\sqrt{2k+1}}{2^k}$

(ii) $\operatorname{cosec}^2 \frac{\pi}{2k+1} + \operatorname{cosec}^2 \frac{2\pi}{2k+1} + \ldots + \operatorname{cosec}^2 \frac{k\pi}{2k+1} = \frac{2}{3} k(k+1)$

and simplify

(iii) $\cot^2 \frac{\pi}{2k+1} + \cot^2 \frac{2\pi}{2k+1} + \ldots + \cot^2 \frac{k\pi}{2k+1}$

ANSWER. (i) From the original definition of P(x) in Q.724, $\frac{P(x)}{x}$ is a polynomial whose value at x = 0 is $\binom{2k+1}{1}\left(1-0^2\right)^k - 0 = 2k+1$. From the factorisation of P(x) proved above, $\frac{P(x)}{x}$ evaluated at x = 0 is equal to

$$(-1)^k 2^{2k}\left(-\sin^2 \frac{\pi}{2k+1}\right)\left(-\sin^2 \frac{2\pi}{2k+1}\right) \ldots \left(-\sin^2 \frac{k\pi}{2k+1}\right)$$

$$= (-1)^{2k} 2^{2k}\left(\sin \frac{\pi}{2k+1} \ldots \sin \frac{k\pi}{2k+1}\right)^2.$$

Equating these and taking the positive square root yields

$$2^k \sin\frac{\pi}{2k+1} \sin\frac{2\pi}{2k+1} \dots \sin\frac{k\pi}{2k+1} = \sqrt{2k+1}$$

(ii) Replacing x by $\frac{1}{y}$ and then multiplying through by y^{2k+1} in each of the expressions for $P(x)$ in Q.724 yields

$$y^{2k+1}P\left(\frac{1}{y}\right) = \binom{2k+1}{1}\left(y^2-1\right)^k - \binom{2k+1}{3}\left(y^2-1\right)^{k-1} + \binom{2k+1}{5}\left(y^2-1\right)^{k-2} \dots$$

$$\dots + (-1)^k\binom{2k+1}{2k+1}$$

$$= (-1)^k 2^{2k}\sin^2\frac{\pi}{2k+1} \dots \sin^2\frac{2\pi}{2k+1}\left(\operatorname{cosec}^2\frac{\pi}{2k+1} - y^2\right)$$

$$\left(\operatorname{cosec}^2\frac{2\pi}{2k+1} - y^2\right) \dots \left(\operatorname{cosec}^2\frac{k\pi}{2k+1} - y^2\right).$$

Equating the coefficients of y^{2k-2} in these polynomials yields

$$-\binom{2k+1}{1}\binom{k}{1} - \binom{2k+1}{3} = -2^{2k}\sin^2\frac{\pi}{2k+1} \dots \sin^2\frac{k\pi}{2k+1}$$

$$\left(\operatorname{cosec}^2\frac{\pi}{2k+1} + \dots \operatorname{cosec}^2\frac{k\pi}{2k+1}\right)$$

$$\therefore\ (2k+1)\left[k + \frac{2k(2k-1)}{6}\right] =$$

$$(2k+1)\left[\operatorname{cosec}^2\frac{\pi}{2k+1} + \dots + \operatorname{cosec}^2\frac{k\pi}{2k+1}\right]$$

(using the result of (i))

$$\therefore\ \operatorname{cosec}^2\frac{\pi}{2k+1} + \dots + \operatorname{cosec}^2\frac{k\pi}{2k+1} = \frac{3k+k(2k-1)}{3} = \frac{2}{3}k(k+1)$$

(iii) Since $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$, the result of (ii) gives

$$\left(1+\cot^2\frac{\pi}{2k+1}\right) + \dots + \left(1+\cot^2\frac{k\pi}{2k+1}\right) = \frac{2}{3}k(k+1)$$

whence $$\cot^2\frac{\pi}{2k+1} + \dots + \cot^2\frac{k\pi}{2k+1} = \frac{2}{3}k(k+1) - k$$

$$= \frac{1}{3}k(2k-1).$$

Correct solution from F. Antonuccio.

Q.726. Using (ii) and (iii) in Q.725 deduce that if $S_k = \frac{1}{1^2} + \frac{1}{2^2} \dots + \frac{1}{k^2}$

$$\frac{\pi^2}{6}\left[1 - \frac{6k+1}{(2k+1)^2}\right] < S_k < \frac{\pi^2}{6}\left[1 - \frac{1}{(2k+1)^2}\right]$$

and find the "limit sum" of the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots . + \frac{1}{k^2} + \ldots$$

ANSWER. Setting $f(\theta) = \theta - \sin\theta$ and $g(\theta) = \tan\theta - \theta$ we have

$f(0) = g(0) = 0$, and $f'(\theta) = 1 \cos\theta > 0$, $g'(\theta) = \sec^2\theta - 1 > 0$ in $0 < \theta < \pi/2$.

Hence $f(\theta) > 0$ and $g(\theta) > 0$ in $0 < \theta < \pi/2$.

Thus $\sin\theta < \theta < \tan\theta$ in $0 < \theta < \pi/2$.

and $\cot\theta < \frac{1}{\theta} < \operatorname{cosec}\theta$ in $0 < \theta < \pi/2$.

Now the results of Q.725 yield

$$\frac{k(2k-1)}{3} = \sum_{r=1}^{k} \cot^2 \frac{r\pi}{2k+1} < \sum_{r=1}^{k} \frac{(2k+1)^2}{\pi^2}\frac{1}{r^2} < \sum_{r=1}^{k} \operatorname{cosec}^2 \frac{r\pi}{2k+1} = \frac{2}{3}k(k+1).$$

Multiplying through by $\frac{\pi^2}{(2k+1)^2}$

$$\frac{\pi^2}{6}\,\frac{4k^2-2k}{4k^2+4k+1} < S_k < \frac{\pi^2}{6}\,\frac{4k^2+4k}{4k^2+k+1}$$

$$\frac{\pi^2}{6}\left[1 - \frac{6k+1}{(2k+1)^2}\right] < S_k < \frac{\pi^2}{6}\left[1 - \frac{1}{(2k+1)^2}\right]$$

Since $\frac{6k+1}{(k+1)^2}$ $(< \frac{12}{2k+1})$ becomes negligibly small as $k \to \infty$, we obtain

$\frac{\pi^2}{6} \le \lim_{k\to\infty} S_k \le \frac{\pi^2}{6}$ and the limit sum has the value $\frac{\pi^2}{6}$.

Correct solution from F. Antonuccio.

Q.727. When a certain polynomial, $P(x)$, is divided by $(x -3)$ the remainder is 5. When $P(x)$ is divided by $(x + 1)$ the remainder is -3. Find the remainder when $P(x)$ is divided by $x^2 - 2x - 3$.

ANSWER. From the first statement, $P(x) = (x-3)Q(x) + 5$ where $Q(x)$ is the quotient when $P(x)$ is divided by $x - 3$. Let $Q(x) = (x+1)R(x) + r$. By the remainder theorem

$$-3 = P(-1) = (-1-3)Q(-1) + 5$$
$$= -4r + 5.$$

Hence $r = 2$ and $P(x) = (x-3)[(x+1)R(x)+2] + 5$

$$= (x^3-2x-3)R(x) + (2x-1).$$

Thus the remainder when $P(x)$ is divided by $x^2 - 2x - 3$ is $2x - 1$.

Correct solution from F. Antonuccio.

Q.728. Find all solutions of the simultaneous equations

$$x_1 + x_3 = xx_2;\ x_2 + x_4 = xx_3;$$

$$x_3 + x_5 = xx_4;\ x_4 + x_1 = xx_5;\ x_5 + x_2 = xx_1.$$

ANSWER. We are to find all values of the unknowns $x, x_1, x_2, \ldots, x_5$ for which all five equations hold.

Adding all equations yields

$$2(x_1 + \ldots + x_5) = x(x_1 + \ldots + x_5)$$

Hence for all solutions, either $x = 2$, or $x_1 + x_2 + x_3 + x_4 + x_5 = 0$

From the first equation we can express

$$x_3 = xx_2 - x_1 \qquad (1)$$

Substituting this into the second , $x_4 = xx_3 - x_2$, gives

$$x_4 = (x^2-1)x_2 - xx_1. \qquad (2)$$

Substituting (1) and (2) into the next equation yields

$$x_5 = (x^3-2x)x_2 - (x^2-1)x_1 \qquad (3)$$

Similarly $$x_1 = (x^4-3x^2+1)x_2 - (x^3-2x)x_1 \qquad (4)$$

and $$x_2 = (x^5-4x^3+3x)x_2 - (x^4-3x^2+1)x_1. \qquad (5)$$

Case 1 $x = 2$.

Each of (4) and (5) simplifies to $x_1 = x_2$. Then we obtain $x_3 = x_1 = x_4 = x_5$ from (1), (2) and (3).

i.e. $$x_1 = x_2 = x_3 = x_4 = x_5 = \alpha \text{ (arbitrary)},\ x = 2$$

gives one family of solutions.

Case 2 $x_1 = x_2 = 0$.

We then have $x_3 = x_4 = x_5 = 0$ from (1), (2) and (3), where x has any value. This gives another obvious family of solutions.

Case 3 If x_1 and x_2 are not both zero, (4) and (5) yield

$$x_1 : x_2 = (x^4-3x^2+1) : (x^3-2x+1) = (x^5-4x^3+3x-1) : (x^4-3x^2+1)$$

$$\therefore (x^4-3x^2+1)^2 = (x^3-2x+1)(x^5-4x^3+3x-1).$$

This simplifies to $x^5 - 5x^3 + 5x - 2 = 0$

and then to
$$(x-2)(x^4+2x^3-x^2-2x+1)$$
$$= (x-2)(x^2+x-1)^2 = 0.$$

$\therefore$ If $x \neq 2$, we must have $x^2 + x - 1 = 0$, $x = \frac{-1\pm\sqrt{5}}{2}$.

Let β denote either of these two numbers. Using (1), (2) and (3) we now obtain the remaining solutions:

$$x = \beta,\ x_3 = \beta x_2 - x_1,\ x_4 = -\beta x_2 - \beta x_1$$
$$x_5 = -x_2 + \beta x_1$$

where x_1 and x_2 may be chosen arbitrarily.

Q.729. A set of cups is arranged in a rectangular array of m rows and n columns, and a random number of beans is placed in each cup (no cup being left empty). The following operations are permitted.

(1) One bean is taken from every cup in a row. (This is not possible obviously if some cup in the row is already empty).

(2) The number of beans in every cup in any column is doubled. Is it always possible to perform these operations repeatedly in such way that all cups are eventually emptied?

ANSWER. It is possible. We shall use R_i to denote the ith row of cups, and C_j to denote the jth column. Op (1,i) means the operation (1) performed on R_i and Op(2,j) means the operation (2) performed on C_j.

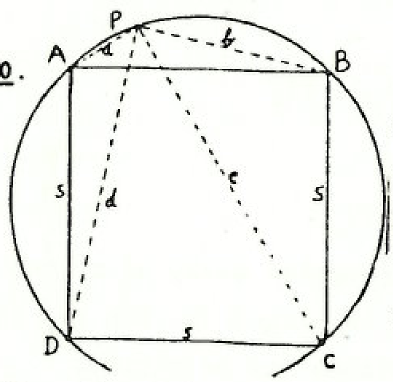
We shall first show that it is possible to empty every cup in R_m. If all cups in R_m contain the same number of beans, this can obviously be accomplished by Op(1,m) only. Otherwise:-

A Perform Op (2, j) for each j (if any) such that the cup in R_m and C_j contains just one bean.

B Perform Op(1,m).

The sequence A, B will decrease by 1 the number of beans in any cup in R_m which contains more than one bean. If the sequence is repeated sufficiently often, we will eventually have 1 bean in each cup in R_m. Then Op (1,m) empties all of them.

Note that no other cup in the array has fewer beans than it had originally, so no other cups are now empty. The whole process can be repeated for R_{m-1}, then R_{m-2} until every row has been emptied. (Note that once a row has been emptied, the cups in it remain empty after all further applications of Op(1,i) or Op(2,j)).



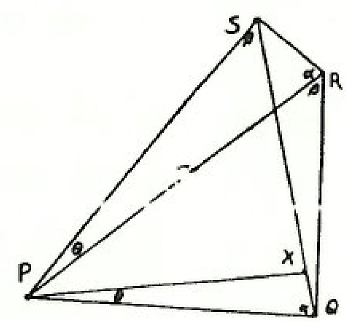
Q.730. In the figure, A B C D is a square, and P is a point on the arc AB of the circumcircle. The distances of P from A, B, C and D are denoted by a, b, c and d respectively. Show that

$(\sqrt{2} + 1)(a + b) = d + c$

and that $a - b = (\sqrt{2} + 1)(d - c)$.

ANSWER. In this and the next problem, we shall make repeated use of the following theorem:-

If PQRS is a cyclic quadrilateral then PQ.RS + PS.QR = PR.QS. (One method of proof constructs a point X on QS such that $Q\hat{P}X = R\hat{P}S$.



Then it is not difficult to prove that

$$\Delta RPS \sim \Delta QPX$$

and also that $\Delta SPX \sim \Delta RPQ$

From these we obtain respectively

$$PQ.SR = RP.QX$$

and $PS.QR = RP.SX$.

Addition of these establishes the desired result.)

Apply the theorem to the quadrilateral ADCP, obtaining

$$a.s + c.s = d(\sqrt{2}s) \qquad (1)$$

where s is the side length of the square ABCD and $\sqrt{2}s$ is the length of each diagonal. Similarly, from quadrilateral PDCB

$$b.s + d.s = c(\sqrt{2}s) \qquad (2)$$

Add (1) and (2), and cancel the factor s

$$(a+b) + (c+d) = \sqrt{2}\ (c+d)$$

Transposing the term (c+d) to the R.H.S. and multiplying through by $(\sqrt{2} + 1)$ gives the first result to be proved.

Now from quadrilateral PACB we have

$$a.s + b.\sqrt{2}s = c.s \qquad (3)$$

From equation (3) subtract $(\sqrt{2}+1)$ times equation (2)

$$a.s + b\sqrt{2}s - (\sqrt{2}+1)b.s - (\sqrt{2}+1)d.s = c.s - \sqrt{2}(\sqrt{2}+1)c.s.$$

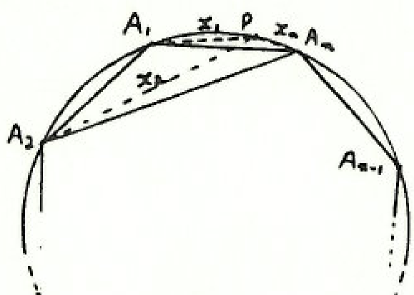
Cancel the factor s, and simplify

$$a - b - (\sqrt{2}+1)d = - (\sqrt{2}+1)c.$$

The second result to be proved follows immediately.

Correct solution from F. Antonuccio.

<u>Q.731</u>.

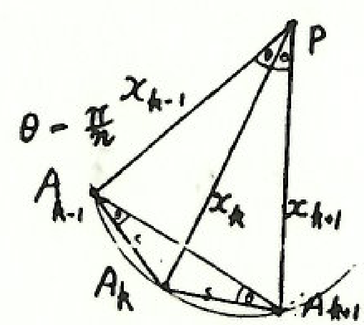


Generalize the first result in Q. 730:
Let P lie on the arc A_1A_n of the circumcircle of a regular polygon $A_1A_2 \ldots A_n$. Let $x_1, \ldots x_n$ denote the distances of P to $A_1, \ldots A_n$ respectively.

Show that $x_2 + x_3 + \ldots + x_{n-1} = \dfrac{\cos\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}(x_1+x_n)$

<u>ANSWER</u>. Applying the theorem used in the previous answer to the quadrilateral $PA_{k-1}A_kA_{k-1}$ gives

$$x_{k-1}.s + x_{k+1}.s = x_k\ 2s\cos\frac{\pi}{n}$$



where s is the side length of the polygon and $2s\cos\frac{\pi}{n}$ is the distance $A_{k-1}A_{k+1}$

(Note $A_k\hat{A}_{k-1}A_{k+1} = \frac{\pi}{n} = A_{k-1}\hat{A}_{k+1}A_k$)

Cancel s and add the results for k = 2, 3, ... k-1. After some obvious simplifying this yields:

$$(x_1+x_n) - (x_2+x_{n-1}) = (x_2+x_3+\ldots+x_{n-1})(2\cos\frac{\pi}{n} - 2) \qquad (1)$$

Now apply the theorem to the quadrilaterals $PA_1A_2A_n$ and $PA_1A_{n-1}A_n$, obtaining

$$x_1.(2\cos\frac{\pi}{n} s) + x_n.s = x_2.s$$

and $$x_1.s + x_n(2\cos\frac{\pi}{n}.s) = x_{n-1}.s.$$

Cancel s, and add to obtain

$$(x_1+x_n)(2\cos\frac{\pi}{n} + 1) = x_2 + x_{n-1} \qquad (2)$$

Substitution of this expression for $x_2 + x_{n-1}$ into the L.H.S. of (1) gives

$$-2\cos\frac{\pi}{n}(x_1+x_n) = (x_2+x_3 + \ldots + x_{n-1})(2\cos\frac{\pi}{n} - 2)$$

Cancel -2 and divide through by $1 - \cos\frac{\pi}{n}$ to obtain the result stated.

Correct solution from F. Antonuccio.