THE U.N.S.W. 27TH SCHOOL MATHEMATICS COMPETITION 1988

QUESTIONS AND SOLUTIONS

JUNIOR DIVISION

1. You are given 6 balls, which appear to be identical except that 1 is white and 5 are black. You are also told that the white and 4 of the blacks are the same weight, while the remaining black is another weight. All you have available is a balance (i.e. a pair of scales) which can be used to compare the weights of two sets of balls, but cannot be used to find their actual weights.
Show how to work out which is the different weight black ball in just two weight comparisons.

Solution (one possible solution)

Denote the balls by W, B_1, B_2, B_3, B_4, B_5.

Compare WB_1 with B_2B_3 -

if same weight, compare WB_1 with B_2B_4:

- if same weight then B_5 is the different weight ball
- if WB_1 is heavier then B_4 is light
- if WB_1 is lighter then B_4 is heavy

if WB_1 is heavier, compare WB_4 with B_1B_2:

- if same weight then B_3 is light
- if WB_4 is heavier then B_2 is light
- if WB_4 is lighter then B_1 is heavy

if WB_1 is lighter, compare WB_4 with B_1B_2:

- if same weight then B_3 is heavy
- if WB_4 is heavier then B_1 is light
- if WB_4 is lighter then B_2 is heavy

Senior: The probability that you can determine whether the different weight ball is heavy or light is 4/5 .

2. In the following sum, each different letter stands for a different non-zero decimal digit and ONE is a prime. What number is THE RIGHT ONE?

ONE + ONE + ONE + TWO + THREE = EIGHT

Solution

Write the carries as the lowest line in the sum.

	ONE		8N7
fig. 1	ONE	fig. 2	8N7
	ONE		8N7
	TWO		6W8
	THREE		69R77
	1cba		1cb3
	EIGHT		7IG96

$E = T + 1$, so $E \neq 1$.

ONE is prime, so $E \neq 2,4,5,6,8$.

$4E + O = T + 10a$, so $O = T + 10a - 4E$.

If $E = 3$ then $T = 2$, $4E = 12$ and $O = 10$, impossible.

If $E = 9$ then $T = 8$, $4E = 36$ so $O = 8$, but $T = 8$, impossible.

Hence $E = 7$, $T = 6$, $4E = 28$, $O = 8$, $a = 3$.

Also $H + c = 10 + I$ and $c \leq 4$ so $H \geq 6$ and $H = 9$.

We are now at figure 2.

$N \neq 1,3,4$ as 817 is divisible by 19, 837 by 3, and 847 by 7, so $N = 2$ or 5.

Now $3N + W + 10 = 10b + 9$, $30 + R + b = 10c + G$,

and $9 + c = I + 10$, so $I = c - 1$.

If $N = 2$ then $W = 3$, $b = 1$, $c = 3$ and $I = 2$, but $N = 2$, impossible.

So $N = 5$, then $W = 4$, $b = 2$, $c = 3$, $I = 2$ and $R + 2 = G$

so $R = 1$, $G = 3$ and we are finished.

Thus 1 = R, 2 = I, 3 = G, 4 = W, 5 = N, 6 = T, 7 = E, 8 = O, 9 = H

and THE RIGHT ONE = 697 12396 857.

3. One day, Albert from Anastasia and Associates Attorneys and Betty from Bartholomew and Brothers Barristers leave at the same time to deliver

messages to the other legal firm. They follow the same route, walking at constant but different speeds and pass each other when Albert has walked A metres. After delivering their messages, and each waiting 10 minutes for a reply, they return to their own firm at the same speeds as before and over the same route, this time passing when Betty is still B metres from home. How far did each walk? If $A < B$ who walks the faster?

Solution

Suppose x = distance between A and B and that Albert walks at u m/s, Betty at v m/s. The 10 minutes is irrelevant, so we ignore it.
At the first time they meet

$$ut_1 = A, \quad vt_1 = x - A, \quad \text{so} \quad \frac{u}{v} = \frac{A}{x-A}$$

At the second time they meet

$$ut_2 = x + B \quad , \quad vt_2 = x + (x-B), \text{ so } \frac{u}{v} = \frac{x + B}{2x - B} .$$

Thus $\frac{A}{x-A} = \frac{x + B}{2x - B}$ so $2Ax - AB = x^2 - Ax + Bx - AB$

i.e. $x(3A - B - x) = 0$ giving $x = 0$ or $x = 3A - B$.

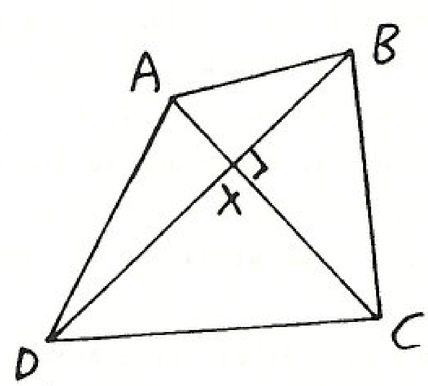
Distance they walk $= 2x = 6A - 2B$ metres.

Also $\frac{v}{u} = \frac{2A-B}{A} = 2 - \frac{B}{A} < 1$ if $A < B$,

so $v < u$ and Albert walks fastest.

4. A quadrilateral shaped frame has pivots at its corners and can freely move. Show that, if its diagonals are ever at right angles, then they are always at right angles.

Solution



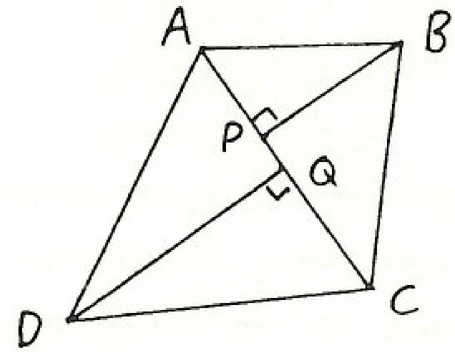
Since $AC \perp BD$ for all arrangements then there must be some relation between the lengths only.
Pythagoras gives

$AB^2 = AX^2 + BX^2$, $BC^2 = BX^2 + CX^2$

$CD^2 = CX^2 + DX^2$, $DA^2 = DX^2 + AX^2$

Hence $AB^2 + CD^2 = BC^2 + DA^2$

or $\quad AB^2 - BC^2 = AD^2 - DC^2$.



In the converse direction we show that if $AB^2 - BC^2 = AD^2 - DC^2$ then $AC \perp BD$.

In triangle ABC with $AB^2 - BC^2$ drop a perpendicular BP from B to AC .

Then $AB^2 - BC^2 = AP^2 + BP^2 - (PC^2 + PB^2) = AP^2 - PC^2$.

Similarly in triangle ADC drop perpendicular DQ to AC,

then $AD^2 - DC^2 = AQ^2 - QC^2$.

Since $AB^2 - BC^2 = AD^2 - DC^2$ then $AP^2 - PC^2 = AQ^2 - QC^2$,

so $AP^2 + QC^2 = AQ^2 + PC^2$. This can only be true if $P = Q$.

Hence BP and DQ = DP are both $\perp$ AC and so $BD \perp AC$.

5. Let A be a number with n decimal digits, and B the number obtained by writing the same digits in reverse order.

If n = 13, prove that in the decimal representation of the sum A + B, There is at least one even digit.

If n = 11 or 12 give examples to show that it is possible for every digit in the sum to be odd.

Solution

For n = 12, e.g.

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   1 2 1 2 1 2 1 2 1 2 1 2
 + 2 1 2 1 2 1 2 1 2 1 2 1
   3 3 3 3 3 3 3 3 3 3 3 3
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For n = 11, e.g.

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    5 0 5 0 5 0 6 0 6 0 6
 +  6 0 6 0 6 0 5 0 5 0 5
  1 1 1 1 1 1 1 1 1 1 1 1
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Similar examples show that every digit in the sum can be odd when n is even and when n = 3, 7, 11, 15 etc.

There must be at least one even digit when n = 5, 9, 13, etc.

The proof for each case is merely an extension of the following proof for n = 5.

$$\begin{array}{r} a_1\ a_2\ a_3\ a_4\ a_5 \\ +\ a_5\ a_4\ a_3\ a_2\ a_1 \\ \hline c_0\ c_1\ c_2\ c_3\ c_4\ c_5 \end{array}$$

where all a_i, c_i are decimal digits and c_0 is 1 or is missing.

If all the c_i are odd, then, as $c_3 = 2a_3$ (even) or $c_3 = 2a_3 + 1$ (odd, with carry 1), we have $a_4 + a_2$ produces a carry. But then $a_2 + a_4$ produces a carry and c_1 comes from $a_1 + a_5 + 1$ and is odd. However $a_5 + a_1$ gives c_5 which is odd and this is impossible if $a_1 + a_5 + 1$ is odd. Hence not all c_i are odd, i.e. there is at least one even digit.

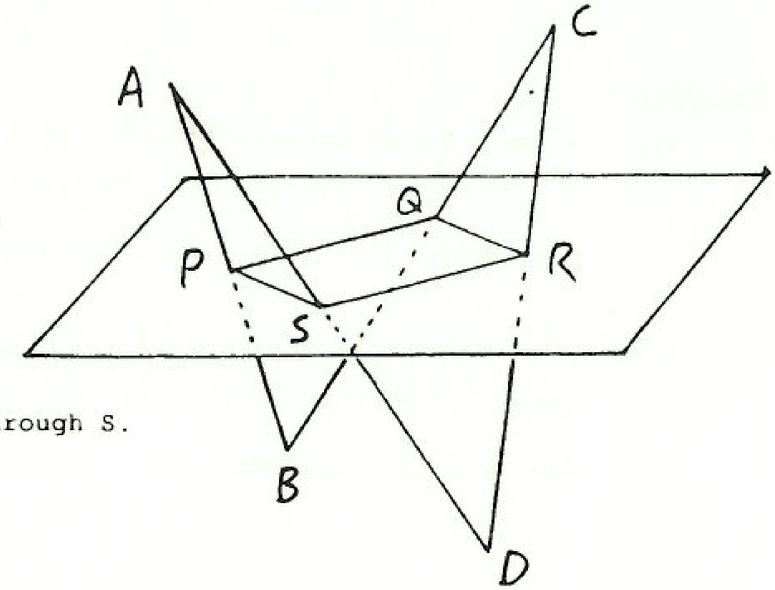
For $n = 13$

$$\begin{array}{r} a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10} a_{11} a_{12} a_{13} \\ +\ a_{13} a_{12} a_{11} a_{10} a_9\ a_8\ a_7\ a_6\ a_5\ a_4\ a_3\ a_2\ a_1 \\ \hline c_0\ c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7\ c_8\ c_9\ c_{10} c_{11} c_{12} c_{13} \end{array}$$

c_7 is odd so there is a carry from $a_8 + a_6$. But then $a_6 + a_8$ produces a carry and c_5 comes from $a_5 + a_9 + 1$ and is odd. Since c_9 is also odd c_9 comes from $a_5 + a_9 + 1$, i.e. $a_{10} + a_4$ produces a carry. But then $a_4 + a_{10}$ produces a carry and c_3 comes from $a_3 + a_{11} + 1$ and is odd. Similarly as c_{11} is odd, a carry comes from $a_{12} + a_2$ and then $a_2 + a_{12}$ produces a carry into c_1 from $a_1 + a_{13}$ i.e. $c_1 = a_1 + a_{13} + 1$ (or $10 + c_1 = a_1 + a_{13} + 1$) and is odd. But $a_{13} + a_1 = c_{13}$ (or $10 + c_{13}$) is odd and both $a_1 + a_{13} + 1$ odd and $a_{13} + a_1$ odd cannot be true. Hence not all c_i are odd and there must be at least one even digit in the sum.

6. Given four points A,B,C,D in space, not all in a plane, show how to find a plane which is the same distance from all four points and has A,C on one side and B,D on the other side.

Solution



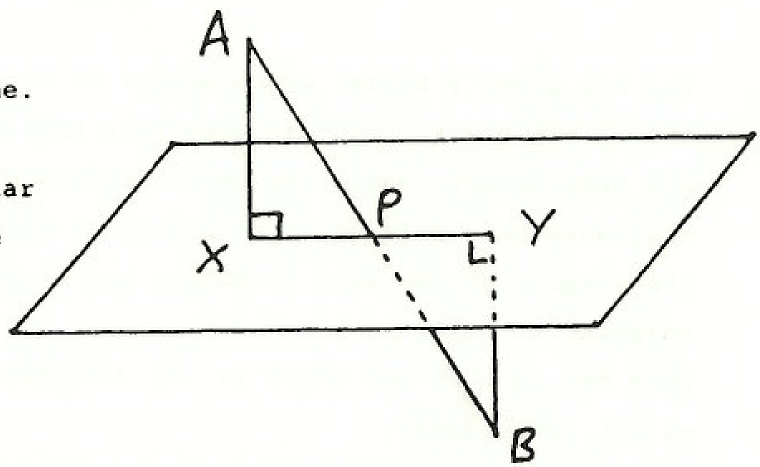
Find the midpoints P,Q,R,S of AB, BC, CD, DA respectively. The required plane is the plane through P, Q, R which we first show passes through S.

PQ∥AC as P,Q midpts,

SR∥AC as S,R midpts,

so PQ∥SR.

Similarly PS∥QR, so that PQRS is a parallelogram and therefore lies in a plane.



We show that the perpendicular distance from A and B to the plane is the same. In the same way it follows that the perpendicular distance from A,B, C and D to the plane is the same.

Drop a perpendicular from A to the plane at X. The plane of AX and AP is perpendicular to the plane and so the perpendicular BY from B to the plane lies in the plane of AX and AP. We now have AP = BP, $\angle APX = \angle BPY$ and $\angle AXP = \angle BYP = 90^\circ$, so triangles AXP and BYP are congruent, and AX = BY as required.

7. If x is a positive number let [x] denote the largest whole number less than or equal to x and $\{x\} = x - [x]$.

For example, [4.16] = 4, {4.16} = 0.16, [4] = 4, {4} = 0.

Find numbers x, y such that

(i) $x^3 - 5[x] = 10$

(ii) $y^3 - 5\{y\} = 10$.

Solution

(i) x^3 must be an integer as $x^3 = 10 + 5[x]$. Also $x^3 \le 10 + 5x$ and $x^3 > 10 + 5(x-1)$ so certainly $2 < x < 3$. Hence $[x] = 2$ and therefore $x^3 = 10 + 5 \times 2 = 20$ and $x = \sqrt[3]{20} = 2.71448...$

(ii) $y^3 = 10 + 5\{y\}$ so $10 < y^3 < 15$ and $\{y\} = y - 2$.
Hence $y^3 = 10 + 5(y-2) = 5y$ and as clearly $y \ne 0$ then $y^2 = 5$ and $y = \sqrt{5} = 2.23606...$

SENIOR

1. You are given 6 balls, which appear to be identical except that 1 is white and 5 are black. You are also told that the white and 4 of the blacks are the same weight, while the remaining black is another weight. All you have available is a balance (i.e. a pair of scales) which can be used to compare the weights of two sets of balls, but cannot be used to find their actual weights.
Show how to work out which is the different weight black ball in just two weight comparisons.
What is the probability that you can also discover if the different weight ball is heavy or light?

Solution

See Junior question 1.

2. A quadrilateral shaped frame has pivots at its corners and can freely move. Show that, if its diagonals are ever at right angles, then they are always at right angles.

Solution

See Junior question 4.

3. A drawer contains some red socks and some green socks, and two socks are removed at random. If the probability that they are both green is exactly one half, what is the smallest number of socks in the drawer?
Suppose you know that there are an even number of red socks in the drawer, what is the smallest number now?
What happens if there is an even number of green socks as well as an even number of red socks?

Solution

Suppose there are r red socks and g green socks. The probability that the first is green $= \frac{g}{r+g}$ and the probability the second is green given that the first is green $= \frac{g-1}{r+g-1}$.

So the probability they are both green is $\frac{g}{r+g} \cdot \frac{g-1}{r+g-1}$ which $= \frac{1}{2}$.

Hence $2g^2 - 2g = r^2 + 2rg + g^2 - r - g$

i.e. $g^2 - 2gr + r^2 - g + r = 2r^2$

i.e. $(g-r)^2 - (g-r) = 2r^2$

i.e. $(g-r)(g-r-1) = 2r^2$.

The smallest solution of this has $g-r = 2$, $r = 1$ and therefore $g = 3$, so there are 4 socks in the drawer (1 red and 3 green).

If r is even, say $r = 2t$, then the equation is

$$(g-2t)(g-2t-1) = 8t^2$$

and the smallest solution of this has $g-2t-1 = 8$, $t = 3$ and therefore $g = 15$, so there are 21 socks (6 red and 15 green).

If g is even also, say $g = 2s$, then the equation is

$$(2s-2t)(2s-2t-1) = 8t^2$$

or letting $s-t = u$, $u(2u-1) = 4t^2$.

We show that this is impossible for integers t, u.

Since $2u-1$ is odd, u is divisible by 4 and can be written as $u = 4v$, giving

$$v(8v-1) = t^2 .$$

Since v and $8v-1$ have no factors in common then $8v-1$ is a square. However there are no squares which are one less than a multiple of 8 so we have a contradiction.

Hence there is no solution when there are even numbers of both red and green socks.

(Notes: if d was a divisor of v and $8v-1$ then it would divide 1, so $d = 1$ is the only common divisor. By considering $(8m)^2$, $(8m+1)^2$, ... $(8m+7)^2$ we see that all squares are of the form $8n$, $8n+1$ or $8n+4$ and never $8n+7$ (i.e. $8(n+1)-1$)).

4. For each positive integer n, determine (with proof) the first digit after the decimal point in the decimal expansion of $\sqrt{n^2 + n + 1}$.

Solution

First some experimentation -

$n =$	1	2	3	4	5	...	1000
$\sqrt{n^2+n+1} =$	1.73	2.64	3.60	4.58	5.56	...	1000.5003.

So it appears that the digit is 7 for $n = 1$,
6 for $n = 2, 3$ and 5 for $n \geq 4$. This is proved as follows.

Since $(n + 0.5)^2 = n^2 + n + 0.25 < n^2 + n + 1$ for every $n \geq 1$ the first digit is at least 5.

Since $(n + 0.6)^2 = n^2 + 1.2n + 0.36 > n^2 + n + 1$ for every $n > 3$ then the first digit is not 6 or greater for $n \geq 4$.

Consequently the first digit is 5 when $n \geq 4$.

5. (i) Let $a(n)$ be the number of positive integers less than or equal to $3332n$ which have a factor in common with 3332.

Show that $\frac{a(n)}{n} = 1988$.

(ii) Let p,q,r be distinct prime numbers and let b(n) be the number of positive integers less than or equal to pqrn which have a factor in common with pqr.

Show that $\frac{b(n)}{n} \neq 1988$.

Solution

(i) $3332 = 2^2.7^2.17$

$\frac{1}{2}$ of the numbers involve no factor 2, of these $\frac{6}{7}$ involve no factor 7, and of these $\frac{16}{17}$ involve no factor 17, so the number of numbers with no factors in common with 3332 is $3332n\ \frac{1}{2}\ \frac{6}{7}\ \frac{16}{17} = 1344n$.

Hence 3332n - 1344n = 1988n numbers do have factors in common with 3332, i.e. a(n) = 1988n.

Thus $\frac{a(n)}{n} = 1988$ as required.

(ii) Similarly

$$b(n) = pqrn - (1 - \frac{1}{p})(1 - \frac{1}{q})(1 - \frac{1}{r})pqrn$$

so $\frac{b(n)}{n} = pqr - (p-1)(q-1)(r-1)$.

If all of p,q,r are odd then (p-1), (q-1), (r-1) are even and b(n)/n is odd, but we want it = 1988.

Hence, say p = 2 and we have $\frac{b(n)}{n} = 1988 = 2qr - (q-1)(r-1)$

But 1988 is divisible by 4 and (q-1), (r-1) are both even, so 2qr is divisible by 4, contradicting the fact that q,r must both be odd.

Hence $\frac{b(n)}{n} \neq 1988$.

6. A,B and C are to fight a 3-cornered pistol duel by firing at their choice of target in turn. A fires first, then B, then C, then A,B,C etc, with a person dropping out when hit, until only one person remains.

When taking careful aim, A has a 30% chance of hitting his target, C has a 50% chance and B never misses. What should A's strategy be?

Repeat the problem when C has only a 40% chance of hitting his target.

Solution

If A shoots at C and hits him, then B will surely shoot A.

If A shoots at B and hits him, then C shoots at A, A shoots at C etc until one is hit. The probability of A surviving is

$$.5 \times .3 + .5 \times .7 \times .5 \times .3 + .5 \times .7 \times .5 \times .7 \times .3 + \ldots\ldots$$

corresponding to (C misses, A hits) or (C misses, A misses, C misses, A hits) etc. This probability

$$= .5 \times .3\ (1 + .5 \times .7 + (.5 \times .7)^2 + (.5 \times .7)^3 + \ldots)$$

$$= .15\ \left(\frac{1}{1 - .35}\right) = \frac{3}{13} = 0.23 \text{ (to 2 fig)}$$

If A hits no-one on his first shot, then B clearly shoots his most dangerous opponent C. A then has another shot at B and hits him with probability 0.3, while if he misses then A gets hit by B. So his probability of surviving is 0.3.

So A's best strategy is to shoot into the air or deliberately miss. Too bad for C.

The best survival strategy of the whole group is for all to miss - but then why fight the duel?

When C's chance is reduced to 40% then A's survival probability if he hits B is $.6 \times .3\ (1 + .6 \times .7 + (.6 \times .7)^2 + (.6 \times .7)^3 + \ldots)$

$$= .18 \left(\frac{1}{1 - .42}\right) = \frac{9}{29} = 0.31 \quad \text{(to 2 fig)}$$

So now his strategy should be to try to hit B.

7. There are $n \geq 4$ points on a circle, numbered 1 to n in some order. Two non-neighbouring points A and B are said to be "linked" if points on at least one of the two arcs between A and B all have numbers smaller than those of both A and B.

Prove that the number of "linked" pairs of points is exactly n-3.

Solution

We use induction.

For $n = 4$ points we have only two cases to consider, the order 1234 around circle or the order 1324. In the first case only 2-4 can be linked, in the second only 3-4 can be linked.

Suppose that for any n point arrangement there are exactly n-3 linked pairs.

Consider an (n+1) point arrangement. By removing the point numbered 1 we have an n point arrangement with all the numbers increased by 1. There are exactly n-3 linked pairs in it. Now put the point numbered 1 back again. The points either side of it can now be linked as they are now non-neighbouring and the arc through 1 is appropriate. So there are (n-3)+1 linked pairs. The number 1 cannot be part of a linked pair and so there are exactly (n+1)-3 linked pairs for the (n+1) point arrangement.
The result now follows by the Principle of Induction.

Q.752 cont'd from bottom p24.

carrier or carriers in that batch. That testing strategy turns out to reduce the average cost per person to about $28.
Is it possible to find a testing strategy which reduces the cost of testing the community to less than $10 per person on average?

A FINAL PARADOX?

A condemned prisoner awaits his fate. He has been told that he is to be executed one morning during the following week. He has also been assured that he will not know the night before that he is to be executed the next morning. In his desperate state his mind plays tricks. He reasons that since he cannot be executed any later than Friday morning he cannot be executed as late as Friday. This is apparent because if he were to be executed on the Friday he would have prior knowledge on the Thursday evening that he is to be executed the next morning. This of course contradicts his jailers' assurance that he wouldn't know the night before. He concludes that if he is to meet the hangman under the stated conditions it must be on a morning of one of the four days Monday, Tuesday, Wednesday or Thursday. Again he figures that it cannot be Thursday for if this were the case he'd know for sure on Wednesday night. He wipes his brow for it now seems he is to be executed during the next three days. But then he realizes he should be safe for the previous argument rules out Wednesday, Tuesday and Monday in turn. Alas, he finds himself at the gallows on Wednesday morning. Desperately he tries to understand where his argument breaks down. Can you help him?