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# SOLUTIONS OF PROBLEMS Q. 732 - Q. 743

**Q. 732.** Let $L$ be a set of $n$ line segments with the property that any three of them can be assembled to form a triangle. A pair of line segments is called "exceptional" if one is more than twice as long as the other.

What is the maximum possible number of exceptional pairs in L?

**ANSWER.** Let $s_1$ be a line segment in $L$ of minimum length, $a$. We shall prove that there cannot be any exceptional pair $(s_2, s_3)$ where neither $s_2$ nor $s_3$ is $s_1$. Suppose on the contrary that $b, c$ are the lengths of $s_2, s_3$ respectively and that $c > 2b$. Then since $a + b \le 2b < c$, there can be no triangle with sides $s_1, s_2, s_3$, contradicting the data.

It follows that the only possible exceptional pairs in $S$ are $(s_1,s_2)$, $(s_1,s_3), \ldots,$ $(s_1,s_n)$, a maximum of $(n-1)$ exceptional pairs. This maximum can certainly be achieved; for example, take $s_1$ of length 1 unit, and $s_2, \ldots, s_n$ all of equal length $c$ units, where $c > 2$.

**Q. 733.** Let three lengths $a,b,c$, and a point $P$ be given. It is desired to construct an equilateral triangle $ABC$ with $P$ as an interior point such that the line segments $PA$, $PB$, $PC$ are respectively $a,b$ and $c$.

(i) Find conditions on $a,b,c$, for the construction to be possible.

(ii) Show how the construction can be performed with straight edge and compass.

**ANSWER**. We may assume without loss of generality that $a \le b \le c$.

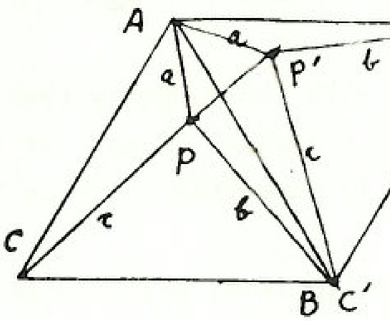



Fig 1

Let ABC be a triangle satisfying the requirements. Rotate it through $60^\circ$ about A (see Fig.1) into the position AB′C′ (where C′ coincides with B); and let P′ be the image of P under the rotation, so that the lengths of P′A, P′B′ and P′C′ are a, b, c respectively.

Since $P\hat{A}P' = 60^\circ$ and $^{*}PA = {}^{*}P'A = a$, $\Delta PAP'$ is equilateral; therefore $PP' = a$. Hence $\Delta PBP'$ has sides of lengths a, b and c, and further, $P'\hat{P}B = A\hat{P}B - A\hat{P}P' < 180^\circ - 60^\circ = 120^\circ$.

Hence our first answer for (i) is that a necessary condition on a,b,c for the construction to succeed is that there should exist a triangle with sides a,b,c whose largest angle is less than $120^\circ$.

(ii) We can construct the whole figure as follows.

Choose any point P′ such that $^{*}P'P = a$. On one side of P′P construct an equilateral triangle PAP′. On the other side construct a triangle PBP′ with $^{*}PB = b$ and $^{*}P'B = c$. AB is now one side of the desired equilateral triangle. The third vertex C may now be constructed.

It is easy to prove that in this triangle ABC the lengths PA, PB, and PC are a,b,c respectively. The first two are immediate from the construction. For the third note that $\Delta BAP'$ is congruent to $\Delta CAP$ since $^{*}BA = {}^{*}CA$, $^{*}AP' = {}^{*}AP$ and $B\hat{A}P' = C\hat{A}P$.

Hence $^{*}CP = {}^{*}BP' = c$ (by construction).

(i) The condition given earlier is now seen to be sufficient as well as necessary. Applying the cosine rule in $\Delta BPP'$

$$B\hat{P}P' < 120^\circ \Rightarrow \frac{a^2 + b^2 - c^2}{2ab} > \cos 120^\circ = -\frac{1}{2}$$

$$\Rightarrow a^2 + ab + b^2 > c^2.$$ This is the required answer to (i) (with the assumption $a \le b \le c$).

Comment: If $a^2 + ab + b^2 = c^2$ the point $P$ lies on the side $AB$ of $\Delta ABC$.

If $a^2 + ab + b^2 < c^2$ but $a + b > c$, a similar construction yields an equilateral triangle with $P$ an exterior point the correct distances from A,B and C. If $a + b < c$ no triangle PBP′ can be constructed, and no equilateral $\Delta ABC$ exists with $P$ either inside or outside having the given lengths PA, PB, PC.

__Q.734__. (i) Let $S$ be a set of rational numbers with the property that the product of every two distinct elements of $S$ is an integer. Show that the product of every $k$ distinct elements of $S$ is an integer for all $k > 2$.

(ii) Show that (i) becomes false if the word "rational" is omitted.

__ANSWER__. (i) First we show that the product of any three of them is an integer. This depends on the following result.

__Lemma__: If $x$ is a rational number such that $x^2 = n$ (n a whole number) then $x$ is an integer.

__Proof__: Suppose $x = \frac{h}{k}$ in lowest terms.

i.e. h and k(>0) are integers without a common factor. If $k > 1$ then $k$ has some prime factor $p$.

$$x^2 = n \Rightarrow \frac{h^2}{k^2} = n \Rightarrow h^2 = nk^2$$

Since $p$ is a factor of $k$ but not of $h$, then $p$ is a factor of $nk^2$, but not of $h^2$. This is a contradiction since $nk^2$ and $h^2$ are the same integer. We are forced to conclude that we must have $k = 1$, and the lemma is proved.
Now let $x = abc$ where $a,b,c \in S$.

Then $x$ is rational and $x^2 = (ab)(ca)(bc)$, being the product of 3 integers, is an integer.
Hence $x$ is an integer, by the lemma.

Finally if $k > 3$, the $k$ distinct elements of $S$ can be grouped in pairs if $k$ is even, or if $k$ is odd the first 3 can be bracketed and the others bracketed in pairs. In either case the product is an integer.

(ii) For a counter example, define $S$ by

$$S = \{ n\sqrt{2} : n = 1,2,3,\dots \}$$

Products of even numbers of elements of $S$ are integral, but a product containing an odd number of factors is $N\sqrt{2}$ for some integer $N$, which is certainly not an integer since $\sqrt{2}$ is irrational.

**Q.735.** Prove that the equation

$$x^{1988} - 2x^{1987} + 3x^{1986} - \dots + 1987x^2 - 1988x + 1989 = 0$$

has no real root.

**ANSWER.** Denote the LHS of the given equation by $F(x)$ and let $G(x) = (x+1)^2 F(x)$. After a little calculation using the formula for summing a G.P. one obtains

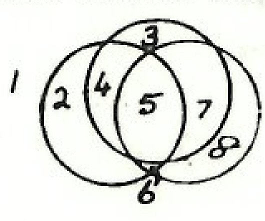
$$G(x) = (x+1)\left[ x\left(\frac{x^{1989}+1}{x+1}\right) + 1989\right] = 0$$

$$= x^{1990} + 1990x + 1989.$$

Since $G'(x) = 1990x^{1989} + 1990$ which vanishes only for $x = -1$, we see that $G(x)$ has a single stationary point, in fact a minimum since $G(x)$ is large and positive for $x$ numerically large. The minimum value is $G(-1) = 0$, so that $G(x) > 0$ for all $x \neq -1$.

Hence $F(x) > 0$ for all $x \neq -1$. It can be immediately verified that $F(-1) \neq 0$. Hence $F(x)$ vanishes for no real value of $x$.

**Q. 736.** One circle divides the plane into 2 regions; two distinct circles give three or 4 regions, depending on their relative position. Three circles can yield 8 regions, but not more.




Find a formula for the maximum number of regions obtainable from $n$ circles, and prove your result.

**ANSWER.** Let $R_n$ be the maximum number of regions obtainable when $n$ circles are drawn. We shall show that $R_{n-1} = R_n + 2n$.

Proof: Consider any diagram with $n$ circles already drawn, containing, say, $R$ regions. Now draw in an $(n + 1)$th circle. If its circumference intersects the boundaries of the regions in $k$ points, these $k$ points divide it into $k$ arcs. Let us draw these arcs one at a time. Each one cuts across a previously existing region creating one more region. Hence altogether an additional $k$ regions result. Since these $k$ intersections result from the $(n + 1)$th circle meeting the $n$ circles previously drawn, and circles cut in at most 2 points we see that $k$ is at most $2n$. ($k < 2n$ if the $(n + 1)$th circle does not intersect a previously drawn circle, or is tangent to it, or if it passes through a point of intersection of two previously drawn circles).

Thus the number of regions after the $(n + 1)$th circle is drawn is at most $R + 2n$.

It follows that $R_{n+1} \le R_n + 2n$. Since we can always draw circles which intersect each other (e.g. draw $(n + 1)$ circles all of the same radius all enclosing a given point $p$) and concurrences of three circles can be avoided by slightly moving one of the offending circles, the inequality may be replaced by equality. i.e. $R_{n+1} = R_n + 2n$ as asserted.

Using this we have

$$R_n = 2(n-1) + R_{n-1} = 2(n-1) + 2(n-2) + R_{n-2} = \dots$$

After $(n-1)$ iterations we obtain

$$R_n = 2(n-1) + 2(n-2) + 2(n-3) + \dots + 2.1 + R_1$$

$$= 2(1+2+ \dots + (n-1)) + 2$$

$$= 2\,\frac{n(n-1)}{2} + 2 = n^2-n+2.$$

such that

$$\sqrt{x + 2 - 4\sqrt{x-2}} + \sqrt{4x - 7 - 4\sqrt{x-2}} = 2 + \sqrt{x - 1 - 2\sqrt{x-2}}$$

(As usual, $\sqrt{y}$ denotes the non-negative square root of y).

**ANSWER.** Put $z = x - 2$

$$\sqrt{z + 4 - 4\sqrt{z}} + \sqrt{4z + 1 - 4\sqrt{z}} = 2 + \sqrt{z + 1 - 2\sqrt{z}}$$

$$\Rightarrow \quad \pm(\sqrt{z} - 2) \pm (2\sqrt{z} - 1) = 2 \pm (\sqrt{z} - 1)$$

where the sign in each case must be chosen to give the positive square root.

If $0 \le z \le \frac{1}{4}$ the signs must be chosen to give

$$(2 - \sqrt{z}) + (1 - 2\sqrt{z}) = 2 + (1 - \sqrt{z})$$

Therefore $2\sqrt{z} = 0$ giving $z = 0$, $x = 2$.

If $\frac{1}{4} \le z \le 1$ the signs must be chosen to give

$$(2 - \sqrt{z}) + (2\sqrt{z} - 1) = 2 + (1 - \sqrt{z})$$

Therefore $2\sqrt{z} = 2$, giving $z = 1$, $x = 3$.

If $1 \le z \le 4$ we similarly obtain

$$(2 - \sqrt{z}) + (2\sqrt{z} - 1) = 2 + (\sqrt{z} - 1)$$

This is true for all z in this range, so all values of x in $3 \le x \le 6$ are solutions.

If $z \ge 4$ we obtain

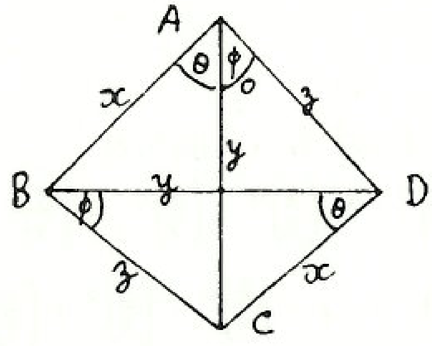
$$(\sqrt{z} - 2) + (2\sqrt{z} - 1) = 2 + (\sqrt{z} - 1)$$

Therefore $2\sqrt{z} = 4$ giving $z = 4$, and $x = 6$.

Hence the given equation is satisfied if $x = 2$ or if x is any number in $3 \le x \le 6$.

**Q. 738.** ABCDE is a tetrahedron having opposite sides of equal length (i.e. AB = CD, AC = BD, AD = BC ). Prove that the faces of the tetrahedron are acute angled triangles.

**ANSWER.**




This depends on the theorem that if three non-coplanar lines are concurrent at A, the sum of any two of the angles $B\hat{A}C$, $B\hat{A}D$ and $D\hat{A}C$ must exceed the third.

Now suppose ABCD is a tetrahedron with the given property, and let x, y, z be the lengths of AB, AC, AD respectively and hence also of CD, DB and BC respectively. Then every face of the tetrahedron has one side of length x, one of length y, and one of length z, so all four faces are congruent triangles. The three angles $B\hat{A}C$, $C\hat{A}D$ and $D\hat{A}B$ are therefore equal to $B\hat{D}C$, $C\hat{B}D$ and $D\hat{C}B$ respectively. Since these are the angles of $\Delta BCD$ their sum is $180^{o}$. Hence if any one of them is $\geq 90^{o}$ it would not be less than the sum of the other two, contradicting the stated theorem. Thus all the faces are acute angled.

**Q. 739.** A random number generator churns out the sequence $x_1, x_2, \ldots, x_n, \ldots$ where each $x_i$ is one of 1, 2, 3, ..., 9 all with equal probability. Let $y_n$ be the product $x_1 x_2 \ldots x_n$. Find the probability that $y_n$ is divisible by 10.

**ANSWER.** We give the following as an example of the notation to be employed:- $P(n;5,\bar{2})$ is the probability that $y_n$ is divisible by 5, but not divisible by 2. Then $P(n;\bar{5}) = \left(\frac{8}{9}\right)^n$ since $y_n$ is not divisible by 5 if and only if every one of the n digits $x_1, x_2, \ldots x_n$ is not a 5, the probability in each case being $\frac{8}{9}$.

Similarly $P(n;\bar{2}) = \left(\frac{5}{9}\right)^n$ (every $x_i \in \{1,3,5,7,9\}$).

and $P(n;\bar{2},\bar{5}) = \left(\frac{4}{9}\right)^n$ (since every $x_i \in \{1,3,7,9\}$

Therefore $P(n;\bar{5},2) = \left(\frac{8}{9}\right)^n - \left(\frac{4}{9}\right)^n$

and $P(n;\bar{2},5) = \left(\frac{5}{9}\right)^n - \left(\frac{4}{9}\right)^n$

Therefore $P(n;\overline{10}) = P(n;\bar{5},2) + P(n;\bar{2},5) + P(n;\bar{5},\bar{2})$

$$= \left(\frac{8}{9}\right)^n - \left(\frac{4}{9}\right)^n + \left(\frac{5}{9}\right)^n - \left(\frac{4}{9}\right)^n + \left(\frac{4}{9}\right)^n$$

Therefore $P(n;10) = 1 - P(n;\overline{10}) = 1 - \left(\frac{8}{9}\right)^n - \left(\frac{5}{9}\right)^n + \left(\frac{4}{9}\right)^n$.

**Q. 740.** A deck contains $n$ cards, of which 3 are kings. This deck is shuffled thoroughly (i.e. until all possible arrangements are equally likely) and then the cards are turned up one by one from the top until the second king appears. If this procedure is repeated many times prove that the average number of cards turned up is likely to be close to $\frac{(n+1)}{2}$.

**ANSWER.** For every arrangement in which the middle king appears in the kth place, there is another arrangement, that obtained by reversing the order of all the cards in the pack, in which it is in the (n+1-k)th place. Thus all the arrangements can be grouped in pairs for which the average number of cards turned up is always $\frac{1}{2}[k + (n+1-k)] = \frac{n+1}{2}$.

Since all arrangements of the cards are equally likely, one would expect that after a sufficiently large number of trials the average of the observed results will be close to the theoretical average.

**Q. 741.** $v, w, x, y, z$ are real numbers such that

$$v + w + x + y + z = 11$$

and $v^2 + w^2 + x^2 + y^2 + z^2 = 25$

Find the largest possible value of $z$.

**ANSWER.** The simplest result concerning the sums of first and second powers of real variables is the following:-

LEMMA: If $x + y = a$ $(>0)$

the smallest possible value of $x^2 + y^2$ is $\frac{a^2}{2}$, obtained only when $x = y = \frac{a}{2}$.

Proof: If $x = \frac{a}{2} + \delta$ then $y = \frac{a}{2} - \delta$

$$\text{and } x^2 + y^2 = \left(\frac{a^2}{4} + a\delta + \delta^2\right) + \left(\frac{a^2}{4} - a\delta + \delta^2\right)$$

$$= \frac{a^2}{2} + 2\delta^2 \text{ which, since } \delta^2 \geq 0$$

for any real $\delta$, is greater than $\frac{a^2}{2}$ except when $x = y = \frac{a}{2}$.

COROLLARY 1 If $x_1 + x_2 + \ldots + x_n = a$ $(>0)$ the smallest possible value of

$x_1^2 + x_2^2 + \ldots + x_n^2$ is $\frac{a^2}{n}$

obtained only when $x_1 = x_2 = \ldots = x_n = \frac{a}{n}$.

Proof: Suppose $x_i \neq x_j$. Replacing $x_i$ and $x_j$ by $x_i' = \frac{x_i + x_j}{2} = x_j'$ leaves the sum of the numbers unchanged, but reduces the sum of the squares, since $x_i'^2 + x_j'^2 < x_i^2 + x_j^2$ by the LEMMA.

Hence the smallest possible value of the sum of the squares is obtained when all of $x_1, \ldots, x_n$ are equal.

COROLLARY 2 If $b < \frac{a^2}{n}$ then there is no solution in real numbers $x_1, x_2, \ldots, x_n$ of

$$\begin{cases} x_1 + x_2 + \ldots + x_n = a. \\ x_1^2 + x_2^2 + \ldots + x_n^2 = b. \end{cases}$$

This follows immediately from Corollary 1.

It follows that $\begin{cases} v + w + x + y = 11 - z \\ v^2 + w^2 + x^2 + y^2 = 25 - z^2 \end{cases}$

has no solution unless

$$25 - z^2 \geq \frac{1}{4}(11-z)^2$$

$$\text{i.e. } 5z^2 - 22z + 21 \leq 0.$$

$$(5z - 7)(z - 3) \leq 0.$$

This is true for $\frac{7}{5} \leq z \leq 3$

Thus the smallest possible value of z is 1.4 and the largest possible value is 3.

(If $z = 3$, the only solution is $v = w = x = y = 2$).

**Q. 742.** If $x$ is a real number, denote by $[x]$ the integral part of $x$ (that is, the largest integer not greater than $x$). Find a positive integer $n$

(i) such that

$$[1^{1/3}] + [2^{1/3}] + [3^{1/3}] + \ldots + [n^{1/3}] = 500$$

(ii) show that

$$[1^{1/2}] + [2^{1/2}] + [3^{1/2}] + \ldots [(m^2 - 1)^{1/2}]$$
$$= \frac{1}{6} m(m-1)(4m+1)$$

for all positive integers m.

**ANSWER.** (i) Note that if $k^3 \leq x < (k+1)^3$ for any whole number k then $[x^{1/3}] = k$. Hence there are $\left((k+1)^3 - k^3\right)$ whole numbers $x$ for which $[x^{1/3}] = k$, viz.

$k^3, k^3+1, \ldots, \left((k+1)^3-1\right)$

In particular there are

$(2^3-1^3)$ whole numbers x for which $[x^{1/3}] = 1$

$(3^3-2^3)$ ........................ $[x^{1/3}] = 2$

Thus $[1^{1/3}] + [2^{1/3}] \ldots + [124^{1/3}]$ (note $124 = 5^3 - 1$)

$= 1 \times (2^3-1^3) + 2 \times (3^3-2^3) + 3 \times (4^3-3^3) + 4 \times (5^3-4^3)$

$= 4 \times 5^3 - 1^3 - 2^3 - 3^3 - 4^3$

$= 400$

There are $6^3 - 5^3 = 91$ whole numbers x such that $[x^{1/3}] = 5$, but to build the sum up from 400 to 500 we need only the first twenty of these:-

$$[125^{1/3}] + [126^{1/3}] \ldots\ldots\ldots\ldots + [144^{1/3}].$$

Hence the required value of n is 144.

(ii) Similarly there are $(k+1)^2 - k^2$ whole numbers x such that $[x^{1/2}]$ is equal to k, namely $x \in (k^2, k^2+1, \ldots, (k+1)^2-1)$.

Therefore the given sum is

$$S = 1\times(2^2-1^2) + 2\times(3^2-2^2) + 3\times(4^2-3^2)+\ldots+(m-1)\left(m^2-(m-1)^2\right)$$

which simplifies to

$$S = (m-1)\,m^2 - [1^2+2^2+3^2+\ldots+(m-1)^2]$$

Now use the identity $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(which may be proved by mathematical induction, or alternatively by summing the identity $k^2 = \frac{1}{3}[(k+1)^3-k^3-1-3k]$ for all values of k from 1 to n and simplifying).

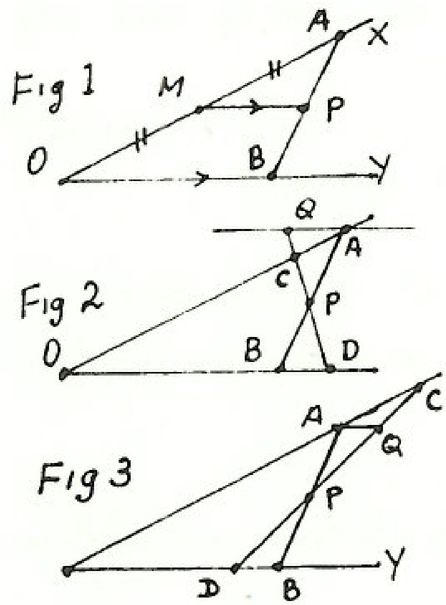
We obtain $S = (m-1)m^2 - \frac{(m-1)m.(2m-1)}{6}$

$$= \frac{(m-1)m}{6}\,[6m - (2m-1)]$$

$$= m\,(m-1)(4m+1)/6$$

Q.743. Given an angle $X\hat{0}Y$ and a point P within its arms, show how to construct points A, B on the arms such that APB is straight and the triangle ΔAOB is of minimum area.

**ANSWER.**




Construct PM parallel to OY meeting OX at M. Construct A on OX such that OM = MA. Produce AP to cut OY at B. Note that *AP = *PB.

To prove that ΔADB is of minimum area, consider any other possible lines through P such as CD in Fig.2 or in Fig.3. Let the line through A parallel to OX cut this line (produced if necessary) at Q. Then the triangles ΔPAQ and ΔPBD are congruent.

In Fig.2 area ΔPAC < area ΔPBD.

Therefore area ΔCOD = area ΔADB + area ΔPBD - area ΔPAC > area ΔAOB.

In Fig.3, area ΔPAC > area ΔPBD

Therefore area ΔCOD = area ΔAOB + area ΔPAC - area ΔPBD > area ΔAOB.

**Correct solutions to problems have been received from**

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