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GEOMETRICAL PROBABILITY AND THE NOTION OF RANDOMNESS

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In probability theory and in everyday language we use the word 'random' and, for many of us, our first encounter with the word would have been in drawing a name at random from a hat.

Most of us have the same commonsense idea in our minds about what 'random' means and this is reflected in the mathematical definition. As with the names in a hat, the simplest situations involve finitely many possible outcomes; for example, if we select a number at random from the integers from 1 to 10, then we define 'random' to mean that the probability that the selected number will be in any subset of the integers from 1 to 10 equals the number of integers in that subset divided by 10, the total number of integers under consideration. In particular, the probability that a particular integer will be the one selected is 1/10; that is, the integers are all equally likely to be selected.

The definition in the finite case extends quite naturally to situations involving intervals; for example, if we select a number at random from the interval [0,10], then by 'random' we mean that the probability that the selected number will be in any interval subset of [0,10] equals the length of that interval divided by 10, the length of the interval [0,10]. This definition of 'random' embodies the idea that all points in the interval are equally likely to be selected.

The 'equally likely' definition can easily be extended to two or more dimensions with, for example, the probability attached to selecting a point at random in some subset of a region in the plane with finite but nonzero area being a ratio of areas.

These definitions not only seem reasonable from a commonsense point of view, but the calculation of probabilities is consistent with the mathematical axioms for probability, the most crucial of which insists that the probability attached to a union of distinct regions must equal the sum of their individual probabilities.

The notion of randomness is of course not restricted to the 'equally likely' case and, in an introduction to probability theory, we consider problems concerned with such things as throwing dice, tossing coins and dealing cards. The fundamental observables are the spots turned up on dice, heads and tails on coins and the faces of cards. The possible outcomes are not usually equally likely. However, attaching probabilities to the various possible outcomes in a random sequence of tosses of dice or coins or of hands of cards is not difficult, given some basic assumptions which specify what we mean by 'random'. Common assumptions include, for example, each die being fair in that each side has probability 1/6 (for the usual six-sided die) of turning up each time the die is thrown or that the probability of head turning up on a toss of a coin is some fixed number p between zero and one each time the coin is tossed. Such assumptions imply that tosses (of a die or coin) are independent of one another in the sense that, for example, the probability of a head turning up on a single toss of the coin is always p, regardless of the results of any previous tosses.

Alternatively, suppose for a sequence of tosses of a coin our assumptions are that the first toss will result in a head with probability p, while for subsequent tosses the probability of recording the same result as the previous toss is p. The sequence of outcomes of tosses is still a random sequence, but the tosses are no longer independent and the probabilities for the possible outcomes are not as before; for example, in three tosses the probability of recording head, tail and then head is $p(1-p)p = p^2(1-p)$ in the independence case and $p(1-p)(1-p) = p(1-p)^2$ under the alternative assumptions.

We should mention in passing that the notion of independence is not easily embraced by gamblers: suppose I have a device which tosses a coin and on each toss the probability of a head is known to me and to a group of gamblers to be 1/2. If I told the group that I had just recorded 9 heads in succession (not a rare event-probability 1/512) and asked each gambler his assessment of the chances of head on the tenth toss, I suspect that some would feel that a head was very likely (probability much larger than 1/2), while others would be just as confident that the run of heads would end.

In geometrical probability the fundamental observables are geometrical objects such as points, lines, triangles, arcs on circles and caps on spheres. As in ordinary probability, attaching probabilities to the various possible outcomes associated with random events is not usually difficult, given some basic assumptions which specify what we mean by 'random'. Different asumptions can lead to different probabilities, just as in ordinary probability. However, in ordinary probability the natural assumptions which specify what we mean by 'random' are usually clear, but this is not always the case in geometrical probability, so that apparent paradoxes can be exposed. The explanation is simply that different definitions of 'random' can lead to different answers. This point is easily illustrated by three solutions to a problem of Bertrand ('Calcul des Probabilités', 1907) and three solutions to a problem about random triangles.

First the Bertrand problem, which is to find the probability that a 'random chord' of a circle of radius one has length greater than $\sqrt{3}$. Bertrand's solutions are as follows:

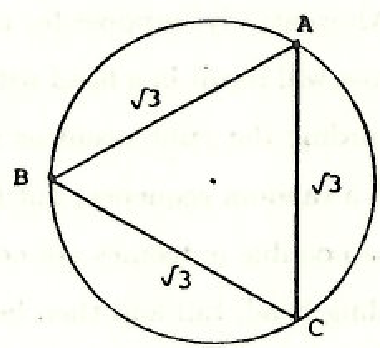


Figure 1

(1) A chord of a circle is uniquely determined by two points on the circle, so suppose that the points which determine the random chord are selected at random on the circle, where by 'at random' we use an equally likely approach and mean that, for each point, the probability that the point lies in any arc of the circle equals the length of the arc divided by 2π, the circumference of the circle. This of course implies that the points are selected independently. Given one of the two points defining the random chord, say A is Figure 1, we can then consider the inscribed equilateral triangle ABC. For a random chord with one endpoint at A to have length at least $\sqrt{3}$, the other endpoint must fall in the arc BC whose length is $2\pi/3$. Thus the

probability that the random chord has length at least $\sqrt{3}$ is $\frac{2\pi/3}{2\pi} = \frac{1}{3}$.

(2) A chord of a circle is also uniquely determined by the point at which a line from the centre of the circle perpendicular to the chord meets the chord. Thus, if E is the centre of the circle, given the point D in Figure 2, there is precisely one chord passing through D and perpendicular to ED. Suppose the point D is chosen at random inside the circle, where by 'at random' we use the equally likely definition for the plane and mean that the probability that the point lies in any region equals the area of that region divided by π, the area inside the circle. For the random chord to have length greater then $\sqrt{3}$ the point D must lie inside a circle of radius $\frac{1}{2}$ (see Figure 3) and hence the probability that the chord has length at least $\sqrt{3}$ is $\frac{\pi/4}{\pi} = \frac{1}{4}$.

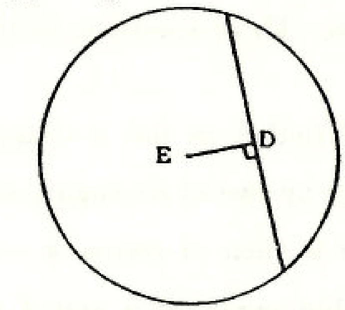
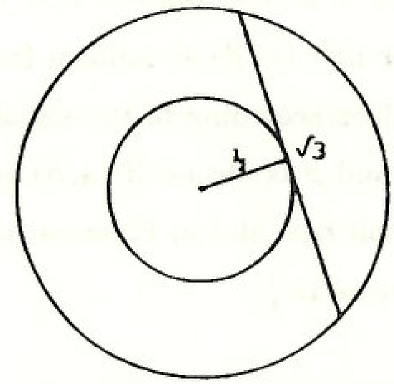


Figure 2



(3) As in Figure 2, a chord is uniquely determined by the point D, but the length of the chord depends only on the length of ED and not on the angle ED makes with any axis. Thus, whatever the direction of ED, suppose its length is selected at random from the numbers between zero and one using the equally likely approach. Then, for the random chord to have length greater than $\sqrt{3}$, the length of the perpendicular from the centre of the circle to the chord must be less than $\frac{1}{2}$, so the probability is $\frac{1}{2}$.

All three solutions are correct of course, the answers being different because the solutions, although based on the 'equally likely' definition of 'random' from ordinary probability notions, are based on three different definitions of a random chord.

The second problem is that of calculating the probability that a random triangle is obtuse. Three solutions to this problem are as follows:

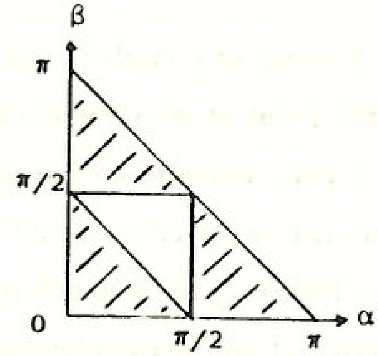


Figure 4

(1) Whether or not a triangle is obtuse is determined by two of its angles, say α and β. The third angle is then of course $\pi - \alpha - \beta$ and the set of possible values of α and β must satisfy $0 < \alpha < \pi, 0 < \beta < \pi$ and $\alpha + \beta < \pi$. Suppose we choose the pair (α, β) at random from the set of possible values according to the equally likely definition for the plane. Then a triangle with angles α and β is obtuse if (α, β) is in the shaded region of Figure 4 and hence, since the four small triangles in Figure 4 have the same area, the probability that a random triangle is obtuse is $\frac{3}{4}$.

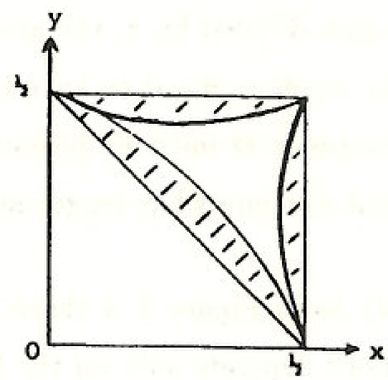


Figure 5

(2) Suppose we consider triangles with a fixed perimeter, which we can take to be one since the probability we are about to calculate is the same whatever the perimeter. Then a triangle is uniquely determined by the lengths, say x and y, of two of its sides. Of course to form a triangle it is necessary to have the length of each side smaller than the sum of the other two; that is, $x < y + 1 - x - y, y < x + 1 - x - y$ and $1 - x - y < x + y$. These inequalities reduce to $x < \frac{1}{2}, y < \frac{1}{2}$ and $x + y > \frac{1}{2}$. Suppose we choose the pair (x, y) at random according to the equally likely definition from the set of possible points in the plane specified by these inequalities. Then a triangle with sides x, y and $1 - x - y$ will be obtuse if the squared length of the longest side is larger than the sum of the squared lengths of the other sides; that is, if $x^2 > y^2 + (1 - x - y)^2$ or $y^2 > x^2 + (1 - x - y)^2$ or $(1 - x - y)^2 > x^2 + y^2$. The region defined by these inequalities inside the set of possible points (x, y) is represented by the shaded area in Figure 5. This area can be shown to equal $3(\frac{1}{8} - \frac{1}{2}\log_e 2)$ and the probability that a triangle is obtuse is then $3(\frac{3}{8} - \frac{1}{2}\log_e 2)/\frac{1}{8} = 9 - 12\log_e 2$ which is

approximately 0.6822.

(3) Another way to form a random triangle is as follows: suppose we let one side of the triangle have some fixed length, z say, and we choose the lengths, say x and y, of the other sides according to the equally likely definition from the region in the plane where $x < z, y < z$ and $x + y > z$. Since the longest side has length z, for the triangle to be obtuse we must have $x^2 + y^2 < z^2$. The region specified by this inequality inside the set of possible points $x, y)$ is represented by the shaded area in Figure 6 and its area can be shown to be $(\frac{\pi}{4} - \frac{1}{2})z^2/\frac{z^2}{2} = \frac{\pi}{2} - 1$, which is approximately 0.5708.

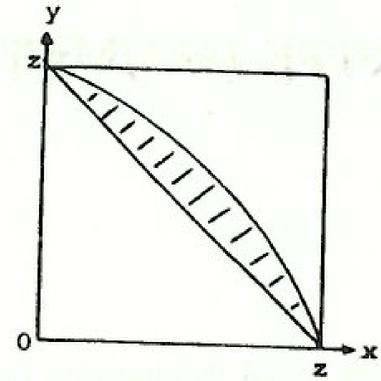


Figure 6

As with the Bertrand problem, all three solutions are correct, but the answers differ because the solutions are based on three different definitions of a random triangle.

Discussion of the notion of randomness and the apparent paradoxes with random chords and triangles has served to introduce the topic 'geometrical probability'. However, the reader should be aware that, although a considerable amount has been written on the subject, very little of it is about paradoxes and rarely are the arguments and calculations so straightforward as here. Many problems involving geometrical probabilities and related quantities have been discussed in the literature and have arisen from many diverse fields including astronomy, biology, crystallography, genetics, medicine, physics and traffic flow theory. These and other fields will no doubt generate further challenging problems in geometrical probability.