SOLUTIONS OF PROBLEMS 744 - 752

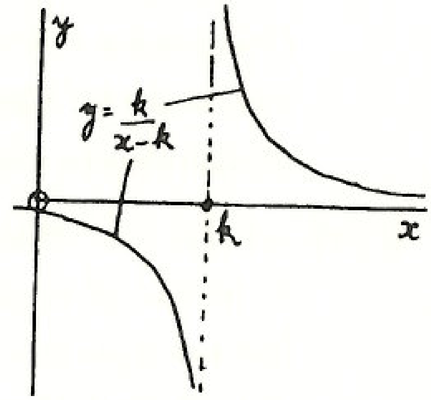
Q.744 Define $f(x) = \sum_{k=114}^{184} \frac{k}{x-k}$

Describe the graph of $f(x)$ and observe that if c is any positive number, the set $\{x : f(x) > c\}$ is the union of 71 intervals.

Let $L(c)$ denote the sum of the lengths of these intervals.

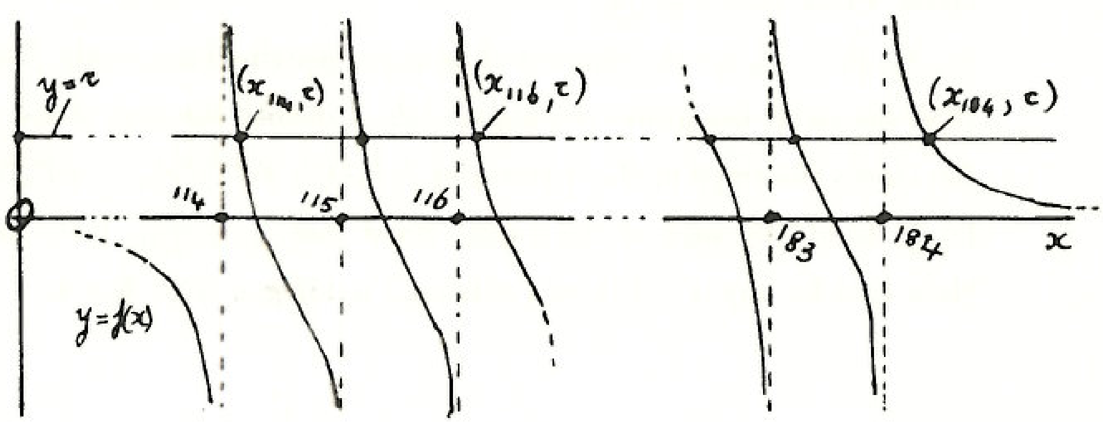
Show that $L(\frac{1988}{336}) = 1788$, and $L(\frac{1788}{336}) = 1988$.

ANSWER: The function $g_k(x) = \frac{k}{x-k}$ $(k \epsilon \mathrm{N})$ has a graph asymptotic to the line $x = k$. It is negative when $x < k$, positive when $x > k$, and is decreasing in both regions since $g'(x) = \frac{-k}{(x-k)^2}$, and it approaches zero as $x \to \pm\infty$.



The function $f(x)$ obtained by summing such functions obtained by taking $k = 114, 115, \cdots, 184$, must then be negative for all $x < 114$, and positive for all $x > 184$. For values of x just exceeding some integer K in the range, the large positive value of $\frac{K}{x-K}$ will dominate the other terms in the sum. A technical way to say this is $\lim_{x \to K+} f(x) = +\infty$ for $K = 114, 115, \cdots, 184$.

Similarly $\lim_{x \to K-} f(x) = -\infty$ for each of these values of K. The sum of decreasing functions is decreasing, and the sum of a finite set of functions tending to 0 at $\pm\infty$ also tends to 0 at $\pm\infty$. Hence the graph of $f(x)$ is as follows:-



On the figure is sketched also the line $y = c$ for some $c > 0$. This line clearly intercepts the graph in one point in each interval $k < x < k+1$; $k \epsilon 114, \cdots, 183\}$ and also at one point in $184 < x$. If we label the x coordinates of those points $x_{114}, x_{115}, \cdots, x_{184}$ we see that $f(x) > c$ in each of the 71 intervals $k < x < x_k$, and nowhere else.

$$\therefore L(c) = \sum_{k=114}^{184} (x_k - k) = \sum_{k=114}^{184} x_k - \sum_{k=114}^{184} k \tag{1}$$

Now $\{x_k : k = 114, \cdots, 184\}$ are the roots of $\displaystyle\sum_{k=144}^{184} \frac{k}{x-k} = c.$

If one multiplies through by $(x - 114)(x - 115)\ldots(x - 184)$ one obtains a polynomial equation.

$$c \prod_{k=114}^{184} (x-k) - \sum_{k=114}^{184} k \left(\prod_{\substack{\ell=114 \\ \ell \neq k}}^{184} (x - \ell) \right) = 0$$

$$\left[c\, x^{71} - \left(c \sum_{k=114}^{184} k \right) x^{70}_{+\ldots} \right] - \left[\sum_{k=114}^{184} k\, x^{70} + \ldots \right] = 0$$

$$c\, x^{71} - (c+1) \left(\sum_{k=114}^{184} k \right) x^{70} + \text{terms of lower degree} \ldots = 0$$

$\sum x_k$, the sum of the 71 roots, is $\displaystyle\frac{c+1}{c} \sum_{k=114}^{184} k$

and from (1)

$$L(c) = \left(\frac{c+1}{c} - 1 \right) \sum_{k=114}^{184} k$$

$$= \frac{1}{c} \sum_{k=114}^{184} k = \frac{1}{c} 71 \times \frac{114 + 184}{2}$$

$$= \frac{1}{c} 71 \times 149$$

$$L\left(\frac{1988}{336}\right) = \frac{336}{1988} \times 71 \times 149 = 1788$$

$$L\left(\frac{1788}{336}\right) = \frac{336}{1788} \times 71 \times 149 = 1988$$

Q.745 A function $f(n)$ is defined for positive integers n in such a way that $f(1) = 1, f(2) = 2$, and if $3^m \leq n < 3^{m+1}$ for a non- negative integer m then $f(3n + k) = k3^m + f(n)$ for $k = 0, 1$, or 2 .
For how many values of n between 1788 and 1988 is $f(n) = n$?

ANSWER: I find that I did not correctly set the question I intended. (The intended question is reset in the new collection of problems). Owing to my error the question as it appears is not particularly interesting since it soon becomes clear that for $n \geq 3$, $f(n) < n$, and hence there are no values of n between 1788 and 1988 for which $f(n) = n$. To prove this, express n in "ternary" notation; i.e. using 3 as the base of the number system.
If $n = a_t3^t + a_{t-1}3^{t-1} + \cdots + a_13 + a_0$ (where each $a_i \epsilon \{0, 1, 2\}$, and $a_t \neq 0$.) then we can calculate

$$f(n) = f(3 \times (a_t3^{t-1} + \cdots + a_1) + a_0)$$

$$= a_03^{t-1} + f(a_t3^{t-1} + \cdots + a_1)$$

and after repeating this sufficiently often we obtain

$$f(n) = a_03^{t-1} + a_13^{t-2} + \cdots + a_{t-1} + f(a_t)$$

$$= a_03^{t-1} + a_13^{t-2} + \cdots + a_{t-1} + a_t.$$

which is clearly less than n , except when $n = 1$ or 2.

Q.746 For any positive integer n let $g(n)$ denote the number of coefficients in the expansion of $(1 + x + x^2 + x^3)^n$ which are odd numbers.
Show that $g(1788) = g(1988)$.

ANSWER: Let $p(x) = \sum a_kx^k$ and $q(x) = \sum b_kx^k$ be polynomials with integer coefficients. We write $p(x) \equiv q(x)$ if a_k and b_k have the same parity (both even or both odd) for all $k \in \mathbb{N}$. For example

$$5 + 3x + 4x^2 + 7x^3 + 2x^4 \equiv 1 + x + x^3 \qquad (1)$$

Note that $p(x) \equiv q(x)$ if all coefficients of $p(x) - q(x)$ are even integers, i.e. if $p(x) - q(x) \equiv 0$.

If all odd coefficients of $p(x)$ are replaced by 1 and all even coefficients by 0, we denote the resulting polynomial by $\bar{p}(x)$ and call it the standard form of $p(x)$. The RHS of (1) is the standard form of the LHS. Obviously $p(x) \equiv \bar{p}(x)$ always. Note that the number of odd coefficients of $p(x)$ is equal to $\bar{p}(1)$; in particular, if $\overline{(1 + x + x^2 + x^3)^n} = f_n(x)$ then $g(n) = f_n(1)$. Observe that,

if $q(x) \not\equiv 0$ $p(x)q(x) \equiv 0$ if and only if $p(x) \equiv 0$ (2)

[Proof:- If $p(x) \equiv 0, p(x) = 2 \times r(x)$ where $r(x)$ has integer coefficients, and $p(x)q(x) = 2 \times r(x)q(x)$. In the other direction suppose $p(x) \not\equiv 0$, and let the terms of highest degree with odd coefficients in $p(x)$ and $q(x)$ be $a_s x^s$ and $b_t x^t$ respectively. Then the coefficient of x^{s+t} in $p(x)q(x)$ is ($a_s b_t$+ even numbers), an odd number, so $p(x)q(x) \not\equiv 0$.]

One consequence of (2) is the following:- If $h(x) \equiv k(x)$ and $r(x) \equiv t(x)$ then $h(x)r(x) \equiv k(x)t(x)$.

Proof : $-$Let $k(x) = h(x)+p(x)$and $t(x) = r(x)+q(x)$.Then $p(x) \equiv 0 \equiv q(x)$, and $k(x)t(x) = (h(x) + p(x))(r(x) + q(x))$

$$= h(x)r(x) + h(x)q(x) + p(x)r(x) + p(x)q(x) \equiv h(x)r(x)$$

since $0 \equiv h(x)q(x) \equiv p(x)r(x) \equiv p(x)q(x)$].

It follows immediately that

$$p(x)^n \equiv \bar{p}(x)^n \tag{3}$$

From the identity $\left(\sum_{i=1}^{m} y_i\right)^2 = (\sum_{i=1}^{m} y_i^2) + 2\sum_{i \neq j} y_i y_j$

one now sees that $\overline{p^2(x)} \equiv (\bar{p}(x))^2 \equiv \bar{p}(x^2)$,

and, by induction on m that $\overline{(p)^{2^m}}(x) = \bar{p}(x^{2^m})$. (4)

Let $\bar{p}(x), \bar{q}(x)$ both be in standard form, and let the degree of $\bar{p}(x)$be s. Then if $s < t$ one sees that

$$\overline{\bar{p}(x)\bar{q}(x^t)} = \bar{p}(x)\bar{q}(x^t) \tag{5}$$

since on multiplying out the RHS, no term has a coefficient other than 0 or 1.

Observe that if $p(x) \not\equiv 0$ and $p(x)q(x) \equiv p(x)r(x)$ then $q(x) \equiv r(x)$, and hence $\bar{q}(x) = \bar{r}(x)$ (6)

This follows from (2) since $p(x)q(x) \equiv p(x)r(x)$ if and only if $p(x)(q(x) - r(x)) \equiv 0$.

One may check that $(1 + x + x^2 + x^3)[(1 + x)(1 + x^4 + x^8 + \cdots + x^{2^m - 4})]$
$\equiv (1 + x^4)(1 + x^4 \cdots + x^{2^m - 4}) \equiv 1 + x^{2^m}$ (7)

Armed with this machinery we may now prove that $g(x)$, the number of odd coefficients in the expansion of $(1 + x + x^2 + x^3)^n$ has the following properties:

$A.$ $g(1) = 4$

$B.$ $g(2^m t) = g(t)$ $\quad m = 0, 1, 2, \cdots$

$C.$ $g(2^m t + 1) = 4g(t)$ $\quad m = 2, 3, \cdots; t$ *odd*

$D.$ $g(2^m t - 1) = 2^{m-2} g(t)$, $\quad m = 2, 3 \cdots; t$ *odd.*

Proof:A is immediate.

B. Letting $f_n(x) = \overline{(1 + x + x^2 + x^3)^n}$ we have already noted that $g(n) = f_n(1)$

By (4) $f_{2^m t}(x) = f_t(x^{2^m})$, whence

$$g(2^m t) = f_{2^m t}(1) = f_t(1^{2^m}) = f_t(1) = g(t).$$

$C.$ $(1 + x + x^2 + x^3)^{2^m t + 1} = (1 + x + x^2 + x^3)^{2^m t}(1 + x + x^2 + x^3)$

$\therefore f_{2^m t + 1}(x) = \overline{f_{2^m t}(x) \quad (1 + x + x^2 + x^3})$

$\qquad = f_t(x^{2^m})(1 + x + x^2 + x^3)$ by (4) and (5)

$\therefore g(2^m t + 1) = f_t(1^{2^m})(1 + 1 + 1 + 1) = 4g(t)$.

$$D. \ (1 + x + x^2 + x^3)^{2^m t} = (1 + x)^{2^m}(1 + x)^{2^m(t-1)}(1 + x^2)^{2^m t}$$

$$\equiv (1 + x^{2^m})(1 + x^2)^{2^m \frac{t-1}{2}}(1 + x^2)^{2^m t} \qquad (8)$$

$$\equiv (1 + x^{2^m})(1 + x^{2^{m+1}})^{t + \frac{t-1}{2}}$$

$\therefore f_{2^m t}(x) = (1 + x^{2^m})\overline{(1 + x^{2^{m+1}})^{t+\frac{t-1}{2}}}$ by the same working as in (5).

$\therefore g(t) = f_{2^m t}(1) = 2 \times \phi(1)$ where $\phi(x) = \overline{(1 + x^{2^{m+1}})^{t+\frac{t-1}{2}}}$ (9)

From (8), using (7),

$(1 + x + x^2 + x^3)^{2^m t} \equiv (1 + x + x^2 + x^3)[(1 + x)(1 + x^4 + \cdots + x^{2^m - 4})]\phi(x)$,

$(1 + x + x^2 + x^3)^{2^m t - 1} \equiv [(1 + x)(1 + x^4 + \cdots + x^{2^m - 4})]\phi(x)$

by (6). Hence

$f_{2^m t - 1}(x) = [(1 + x)(1 + x^4 + \cdots + x^{2^m - 4})]\phi(x)$by (5).

Therefore $g(2^m t - 1) = 2 \times 2^{m-2}\phi(1) = 2^{m-2}g(t)$ from (9).

Finally, using the properties A,B,C,D we now calculate:-

$g(1988) = g(4 \times 497) = g(497) = g(4 \times 124 + 1) = 4 \times g(124) = 4 \times g(31) =$
$4 \times g(2^5.1 - 1) = 4 \times 2^3 g(1) = 4 \times 8 \times 4 = 128$

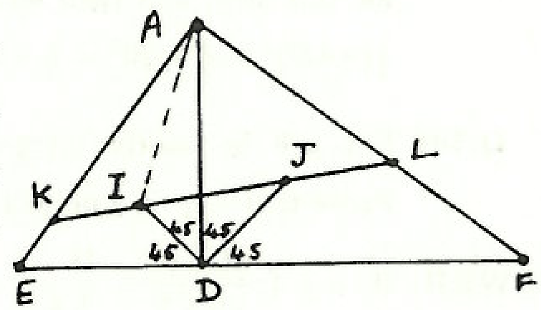
$g(1788) = g(4 \times 447) = g(447) = g(64 \times 7 - 1) = 16g(7)$
$16g(8 \times 1 - 1) = 16 \times 2 \times g(1) = 128$.

Q.747 The triangle ΔABC has side lengths $AB = 25, AC = \frac{65}{3}$, $BC = \frac{70}{3.}$

The internal and external bisectors of $B\hat{A}C$ cut BC at points E, F respectively, and D is the foot of the perpendicular from A to BC . The line passing through the incentres of triangles ΔADE and ΔADF cuts AE at K and AF at L .

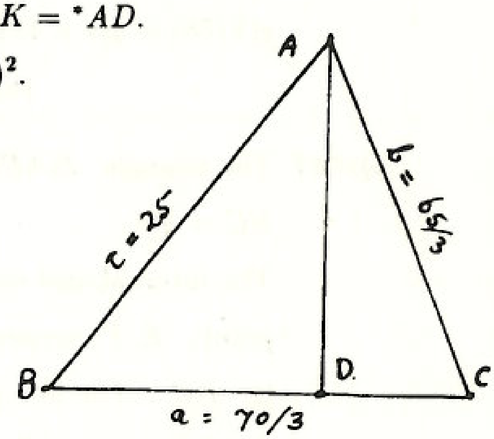
Show that the area of $\Delta AKL = 200$.

ANSWER. In the figure, I and J are the incentres of ΔADE and ΔFDA respectively. It is easy to prove that AE and AF , the internal and external bisectors of $B\hat{A}C$, are at rightangles. Since DI and DJ are bisectors of the right angles $A\hat{D}E$ and $F\hat{D}A$, all four acute angles shown at D are $45°$ angles, and $I\hat{D}J = 90°$. Since ΔEDA is similar to



ΔADF (being equiangular) a clockwise rotation of ΔEDA about D through $90°$ followed by a magnification by the factor $^*AD/^*ED$ will make it coincide exactly with ΔADF, and its incentre I will be mapped onto J, the incentre of ΔADF. It follows that $^*DJ/DI^* =^* AD/^*ED$, and this makes ΔIDJ similar to ΔEDA in view of the rightangles at D. If ΔIDJ is rotated anticlockwise about D through $45°, DI$ and DJ will align with DL and DA respectively, and IJ must be taken into a direction parallel to AE. Hence KL must be inclined at an angle of $45°$ to AE. We can now see that ΔAKL is rightangled isosceles, so $^*AK = {}^*AL$.

The triangles ΔAIK and ΔAID have equal angles at A (AI bisects $E\hat{A}D$) and $A\hat{K}I = A\hat{D}I = 45°$. With the common side AI, the two triangles are congruent, so $^*AK = {}^*AD$.
Thus area $\Delta AKL = \frac{1}{2}{}^*AK.{}^*AL = \frac{1}{2}({}^*AD)^2$.



We can calculate the height $^*AD$ from the given side lengths of ΔABC. Thus $^*AD =$ $2 \times area\Delta ABC/{}^*BC.$ Area ΔABC is given by $\sqrt{s(s-a)(s-b)(s-c)},$ where a, b, c are the side lengths and $s = (a+b+c)/2$. Calculation gives $35 \times 20/3$ for this area, and then we calculate $^*AD = 20$. Finally area $\Delta AKL =$ $\frac{1}{2}(*AD)^2 = \frac{1}{2} \times 20^2 = \frac{1}{2} \times 20^2 = 200.$

Q.748 Let a, b be positive integers such that $q = \dfrac{a^2+b^2}{ab+2}$ is also an integer. Prove that $2q$ is a perfect square.

ANSWER: If $a = 1, \dfrac{a^2+b^2}{ab+2} = \dfrac{b^2+1}{b+2} = b - 2 + \dfrac{5}{b+2}$ which is an integer only if $b = 3$. Then $q = \frac{10}{5} = 2 \times 1^2$. Henceforth we assume that $a \geq 2$, and without loss of generality, that $a \leq b$. Since $\dfrac{2a^2}{a^2+2} < 2$, there is no solution with $a = b$; we let $b = \lambda a - r$ where

λ is an integer greater than 1, and $0 \le r < a$. (i.e., λa is the smallest multiple of a which is not less than b). A little experimenting suggests that $ab+2$ is quite close to λ so we put

$$q = \frac{a^2+b^2}{ab+2} = \lambda + \text{Remainder}.$$

where Remainder $= \dfrac{a^2 - \lambda ar + r^2 - 2\lambda}{\lambda a^2 - ar + 2}$ after some calculation. We shall show that $-1 <$ Remainder < 1.

[Proof that $-1 <$ Remainder < 1. Since $\lambda a^2 - ar + 2 > 0$, $-1 <$ Remainder $\Leftrightarrow -(\lambda a^2 - ar + 2) < a^2 - \lambda ar + r^2 - 2\lambda$

$$\Leftrightarrow 0 < \lambda(a(a-r)-2) + (a^2 - ar) + (r^2+2).$$ This is clear since the R.H.S. is the sum of 3 positive terms. (Remember $a \ge 2, a - r \ge 1$).

Remainder $< 1 \Leftrightarrow a^2 - \lambda ar + r^2 - 2\lambda < \lambda a^2 - ar + 2$

$$\Leftrightarrow 0 < (\lambda - 1)(a^2 + ar - r^2) + (2\lambda + 2) + (\lambda - 2)r^2.$$

which is again clear, since $\lambda \ge 2$.]

It follows that q and λ both being integers which differ by less than 1, must be equal, and that Remainder $= 0$.

$$\therefore a^2 - \lambda ar + r^2 - 2\lambda = 0$$

$$\text{and } q = \lambda = \frac{a^2+r^2}{ar+2}$$

If $r = 0$ this gives $q = \dfrac{a^2}{2} = 2k^2$ where $k = \dfrac{a}{2}$, and if $r = 1$ we have already shown that the only solution is $q = 2 \times 1^2, a = 3$. Otherwise, letting $r = a', a = b'$ our new expression for q is $q = \dfrac{a'^2 + b'^2}{a'b' + 2}$ with $2 \le a' < b'$.

We can then repeat the process obtaining eventually $q = \dfrac{a''^2 + b''^2}{a''b'' + 2}$ with $a'' < a'$. Eventually we must come down to a similar expression for q with $a = 0$ or 1, and we have seen that for such expressions q is the double of a square.

Q.749

(i) ΔPQR is equilateral and α, β, γ are any three angles such that $\alpha + \beta + \gamma = 60°$. Points B and C are constructed as in Figure 1 making $B\hat{R}P = C\hat{Q}P = 60° + \alpha, B\hat{P}R = 60° + \gamma$, and $C\hat{P}Q = 60° + \beta$. Show that $PB \sin\beta = PC \sin\gamma$, and that PB and PC are bisectors of $R\hat{B}C$ and $Q\hat{C}B$ respectively.

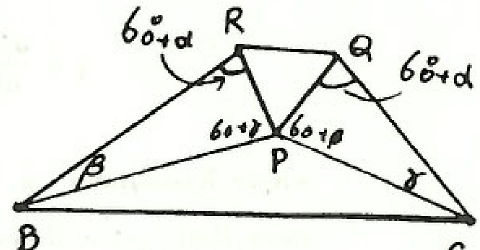


Fig 1

(ii) Let $A'B'C'$ be any triangle with angles $3\alpha, 3\beta, 3\gamma$. Let lines which trisect the angles intersect in pairs at points X, Y, Z as in Figure 2. Prove that ΔXYZ is equilateral.

ANSWER:

(i) In ΔBPR, $P\hat{B}R = 180° - (60 + \alpha) - (60 + \gamma) = 60 - \alpha - \gamma = \beta$. By the sine rule $PB \sin\beta = PR \sin(60 + \alpha) = s\sin(60 + \alpha)$ where s is the side length of the equilateral triangle ΔPQR. Similarly in $\Delta PQC, P\hat{C}Q = \gamma$ and $PC \sin\gamma = PB \sin\beta, \cdots$ (1)

Now $B\hat{P}C = 360° - (60° + \gamma) - 60° - (60° + \beta) = 180° - (\gamma + \beta)$
so that in $\Delta P\hat{B}C, P\hat{B}C + P\hat{C}B = \theta + \phi = \beta + \gamma$ (2)

By the sine rule $PB \sin\theta = PC \sin\phi$.
Hence, using (1), $\dfrac{\sin\theta}{\sin\phi} = \dfrac{\sin\beta}{\sin\gamma}$ (3)

From (2) and (3) it follows that $\theta = \beta$ and $\phi = \gamma$.

[Suppose $\theta > \beta$, then $\phi < \gamma$ from (2), $\sin\theta > \sin\beta$ and $\sin\phi < \sin\gamma$ (since all angles are acute). But this would imply $\frac{\sin\theta}{\sin\phi} > \frac{\sin\beta}{\sin\gamma}$, contradicting (3)].

Thus PB bisects $R\hat{B}C$ and PC bisects $B\hat{C}Q$

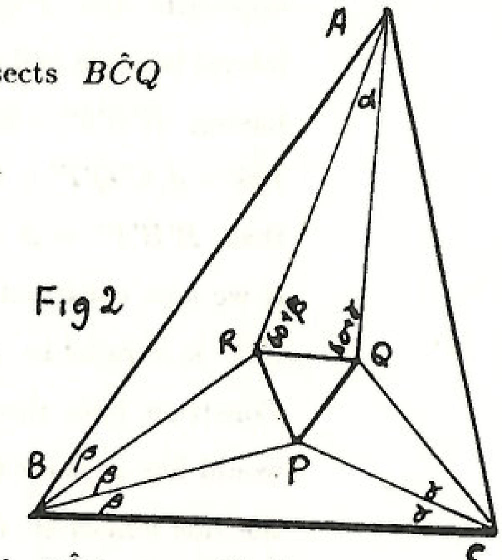
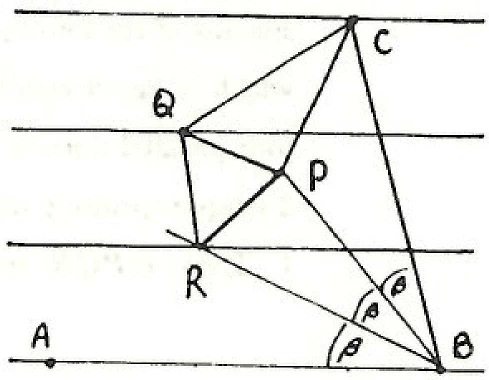


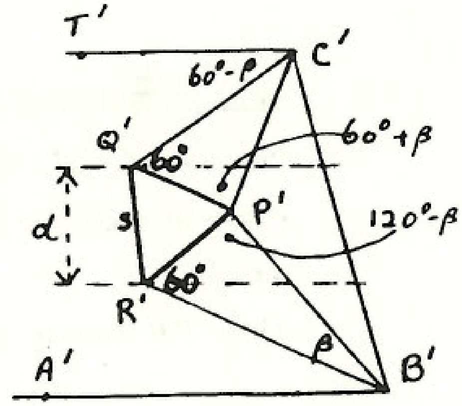
Fig 2

(ii) In Figure 1, if we extend the figure by constructing lines RA and QA making $Q\hat{R}A = 60°+\beta$ and $R\hat{Q}A = 60° + \gamma$, we find immediately that $R\hat{A}Q = \alpha$, and arguing as in (i) we prove that RA, RB, QA, QC are bisectors of angles $Q\hat{A}B, P\hat{B}A, R\hat{A}C$, and $P\hat{C}A$ respectively. (See Figure 2). Thus ΔABC is equiangular with the $\Delta A'B'C'$. If its size is adjusted by the appropriate factor, it can be brought into co-incidence with $\Delta A'B'C'$, and the trisectors of the angles will then coincide in the two triangles, so that ΔXYZ will be in coincidence with the equilateral ΔPQR . Hence the result.



Q.750 A page is ruled with equally spaced parallel lines. Points A, B lie on the fourth line from the top of the page. R is a point on the next line up such that $R\hat{B}A = \beta(< 30°)$, and C is the point on the top line such that $C\hat{B}A = 3\beta$. Q is the point on the second top line (lying on the same side of BC as A and R) such that QC makes an angle $(60° - \beta)$ with the ruled lines. The bisectors of $Q\hat{C}B$ and $R\hat{B}C$ intersect at P (see Figure Prove that ΔPQR is equilateral.

ANSWER: Recognising that this result is very like that in Q.749, but with $\alpha = 0$, it is natural to try a similar indirect approach. Let $P'Q'R'$ be an equilateral triangle and construct B', C' having $B'\hat{R}'P' = 60°, B'\hat{P}'R' = 120° - \beta, C'\hat{Q}'P' = 60°, C'P'Q' = 60° + \beta$. As in Q.749 (i) we can now show that $R'\hat{B}'P' = \beta = P'\hat{B}'C'$ and $Q'\hat{C}'P' = P'\hat{C}'B' = 60° - \beta$.



If we now construct $R'\hat{B}'A' = \beta$ and $Q'\hat{C}'T' = 60° - \beta$ it is clear that $C'T'$ is parallel to $B'A'$ since the angles at B' and C' total $180°$. Construct lines through Q' and R' parallel to $A'B'$ and $T'C'$. We would like to show that these 4 parallel lines are equally spaced. Let s be the side length of $\Delta P'Q'R'$. Since it is easy to see that $Q'R'$ makes an angle of $60° + \beta$ with the parallel through Q', the perpendicular distance, d, between the parallels through Q' and R' is $s\sin(60° + \beta)$.
From $\Delta Q'P'C'$, $Q'C'\sin(60° - \beta) = s\sin(60° + \beta)$ so that the perpendicular distance between the top two parallels is also equal to $Q'C'\sin Q'\hat{C}T = s\sin(60° + \beta) = d$. Similarly, from the sine rule applied to $\Delta R'B'P'$, the spacing of the lower pair of parallels, $R'B'\sin\beta$, is equal to $s\sin(120° - \beta)$, which is again equal to d since $\sin(120° - \beta) = \sin(60° + \beta)$. Thus the four parallel lines in Figure 2 are indeed equally spaced. If the size of Figure 2 is appropriately adjusted it can be made to coincide precisely with Figure 1. Thus ΔPQR must be equilateral.

Q.751 When B. Rainy, the school genius, knocked his calculator off the desk during the maths exam, he discovered that only the algebraic operations $+, \times, -, \div$ were still operational. The only exam question still to be answered required a calculation involving $\log_e 11$. Our hero recollected seeing in a calculus text the theorem

$$log_e\left(\frac{x+1}{x-1}\right) = 2\left[\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5.x^5}+\cdots\right] \text{ whenever } x > 1.$$

After a little calculating he found whole numbers A, B, C such that

$$log_e 11 = A\left(\frac{1}{23}+\frac{1}{3.23^3}+\frac{1}{5.23^5}\right)+B\left(\frac{1}{65}+\frac{1}{3.65^3}\right)-C\left(\frac{1}{485}+\frac{1}{3.485^3}\right)$$

to eight decimal places. He evaluated this on his calculator and completed the examination, obtaining his usual 100%.

Can you find A, B, C and calculate $log_e 11$?

ANSWER:

$$A\left(\frac{1}{23}+\frac{1}{3\times 23^3}+\frac{1}{5\times 23^5}+\cdots\right) + B\left(\frac{1}{65}+\frac{1}{3\times 65^3}\cdots\right)$$

$$-C\left(\frac{1}{485}+\frac{1}{3.485^3}\cdots\right)$$

$$=\frac{A}{2}\log\frac{23+1}{23-1} + \frac{B}{2}\left(\log\frac{65+1}{65-1}\right) - \frac{C}{2}\log\left(\frac{485+1}{485-1}\right)$$

$$=\frac{A}{2}(2\log 2+\log 3-\log 11) + \frac{B}{2}(-5\log 2+\log 3+\log 11)$$

$$-\frac{C}{2}(-\log 2+5\log 3-2\log 11)$$

$$=\left(A-\frac{5}{2}B+\frac{C}{2}\right)\log 2 + \left(\frac{A}{2}+\frac{1}{2}B-\frac{5}{2}C\right)\log 3$$

$$+\left(-\frac{A}{2}+\frac{B}{2}+C\right)\log 11$$

$$=\log 11$$

$$\text{if} \quad \begin{cases} A-\frac{5}{2}B+\frac{C}{2}=0 \\ \frac{A}{2}+\frac{B}{2}-\frac{5}{2}C=0 \\ \frac{-A}{2}+\frac{B}{2}+C=1 \end{cases}$$

These solve to give $A = 48, B = 22, C = 14$.

[The error made in using only the first few terms of the series is in each case approximately equal to the first omitted term, since the terms get small very rapidly. The combined error is of the order of magnitude of

$$A\frac{1}{7\times 23^7} + B\frac{1}{5\times 65^5} + C\frac{1}{5\times 485^5} < \frac{1}{2}\times 10^8]$$

Performing the calculation gives the result

$$log 11 \approx 2.39789527$$

Q.752 In order to locate all the carriers of a disease it has been decided to perform blood tests on everyone in a large community. It is known that the probability that a randomly selected individual has the disease is 1%. Each blood test costs \$100 to perform, but the test is very sensitive and a diseased sample can be detected even if diluted by a factor of several thousand. Instead of just testing each sample separately, it would obviously be cheaper to mix small portions of each of a batch of samples together and test the mixture. For example, if batches of 4 samples were mixed, most of the batches would yield a negative result. Any batch which tested positive would necessitate further tests on the remaining portions of the 4 samples to determine the carrier or carriers in that batch. That testing strategy turns out to reduce the average cost per person to about \$28.

Is it possible to find a testing strategy which reduces the cost of testing the community to less than \$10 per person on average?

ANSWER: Let $p(= 0.01)$ be the probability that a randomly selected sample returns a positive result, and $q = 1 - p$. We consider a testing procedure which starts by mixing small portions of each of a batch of n samples and testing the mixture. Let Z_n be the average number of tests required to locate all the carriers in the batch.

The number of tests is obviously 1 if no-one is a carrier; the probability that this is the case is q^n. If the initial test is positive several more tests will be needed. We let Y_n denote the average number of tests (including the initial one) to locate all carriers in a batch of size n which shows positive on the initial test. Then for every n

$$Z_n = q^n 1 + (1 - q^n)Y_n \tag{1}$$

[Proof: Let some large number N of batches of size n be tested. About Nq^n of the batches have no carrier and require just 1 test. The remaining

$N(1-q^n)$ batches require on average Y_n tests to locate all the carriers. The total number of tests for all N batches is $Nq^n + N(1-q^n)Y_n$. Now divide through by N to get the average number of tests per batch, Z_n.]

Now let $n = 2m$. Our testing procedure mixes small fractions of all $2m$ samples for the first test, A , further small fractions of each of the first m samples into a mixture B and small fractions of each of the other m samples into a mixture C :

We consider four cases:-

Case 1 B and C both "clear" This occurs with probability q^{2m} and only the test on A is needed to test the whole batch.

Case 2 B is clear, but C is not. This occurs with probability $q^m(1-q^m)$. After the test on A shows positive, B is tested and shows clear. Then C must be positive. The average number of tests to find all carriers in C is then $Y_m - 1$ (since we do not need the initial test on the mixture C). In this case, the total number of tests on the batch of size $2m$ is $1(\text{on } A) + 1(\text{on } B) + (Y_m - 1) = Y_m + 1$

Case 3 B is positive, but C is clear. This case occurs with probability $(1-q^m)q^m$. After tests on A and B , there are needed a further $(Y_m - 1)$ tests to locate all carriers in B , and finally one test on C (which reveals that it is clear). This is a total of $Y_m + 2$ tests in this case.

Case 4 B and C both positive. This case occurs with probability $(1-q^m)^2$. In addition to the initial test on A , there will be required on average Y_m tests on both B and C to locate all the positive samples; i.e. a total of $2Y_m + 1$ tests in this case.

From these results it follows that

$$Z_{2m} = q^{2m}1 + q^m(1-q^m)(Y_m+1) + (1-q^m)q^m(Y_m+2) + (1-q^m)^2(2Y_m+1)$$

(This follows from an analysis similar to that in brackets justifying (1) above).

After simplification

$$Z_{2m} = 2Y_m(1 - q^m) + 1 + q^m - q^{2m}$$

$$\text{From (1), } 2Y_m(1 - q^m) = 2Z_m - 2q^m$$

$$\therefore Z_{2m} = 2Z_m + 1 - q^m - q^{2m} \tag{2}$$

Now the average cost per sample is given by $\frac{Z_n}{n}$(times \$100)

From (2)

$$\frac{Z_{2m}}{2m} = \frac{Z_m}{m} + \frac{1}{2m}(1 - q^m - q^{2m}) \tag{3}$$

Thus the average cost per sample is less for batch size $2m$ than for batch size m provided $1 - q^m - q^{2m} < 0$. (When $q = 0.99$ this is so provided $m < 48$). There is another consideration however. Our testing procedure requires the batch to be divided into halves repeatedly (for "positive" batches), and this can be done exactly only if n is a power of 2.

Using (3), with $Z_1 = 1$ and $q = 0.99$ one can calculate $\frac{Z_n}{n}$ for $n = 2, 4, 8, 16\ldots$ etc. obtaining 0.5149, 0.2798, 0.1694, 0.1210, 0.1030, 0.0990, 0.1006, ... respectively.

For $n = 64$, $\frac{Z_{64}}{64} = 0.0990$ and the average cost per sample is $0.0990 \times$ \$100 = \$9.90.