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SPECIAL RELATIVITY

BY WILLIAM TAYLOR

The popularity of the special theory of relativity stems from extraordinary predictions about time, distance, mass, Energy and the nature of space. What follows attempts to connect the concepts of time dilation with the equivalence of mass and energy and the concept of four dimensions. Relativity isn't a "fact", it is a set of ideas which can be used to interpret the results of experiments and observations and to make predictions in a consistant way. So far it has done so with great accuracy.

NOTE: Throughout this essay it is assumed that NO ACCELERATION is going on. Consideration of relativistic effects due to acceleration (or vice versa) are the realm of General relativity and Mach's Principle (which has nothing to do with the speed of sound).

There are two principles

1. The speed of light, in empty space, is the same "c" independent of the state of motion of the observer (and the motion of the source).

2. A principle best stated in several forms

   - Electrodynamic and mechanical effects do not have properties that require the concept of an absolute state of rest.

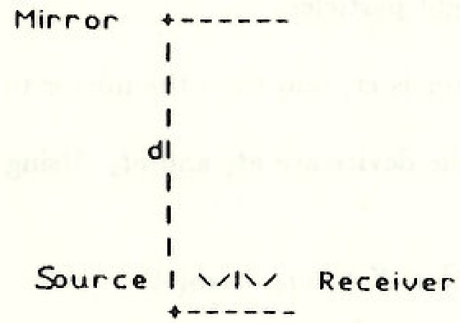
   - The speed of an object cannot be determined IN ANY WAY without external reference.

   - No object is absolutely stationary (This is well known to the highway patrol)

   - The concept of absolute (non-accelerating) motion is invalid.

PART 1 Time and distance measurement

The apparatus I will use is a light source, coupled with a detector, both orientated to use a mirror "d" units away. There is also a stop watch attached to the apparatus that measures the time "t" taken for light to go from the source to the receiver.



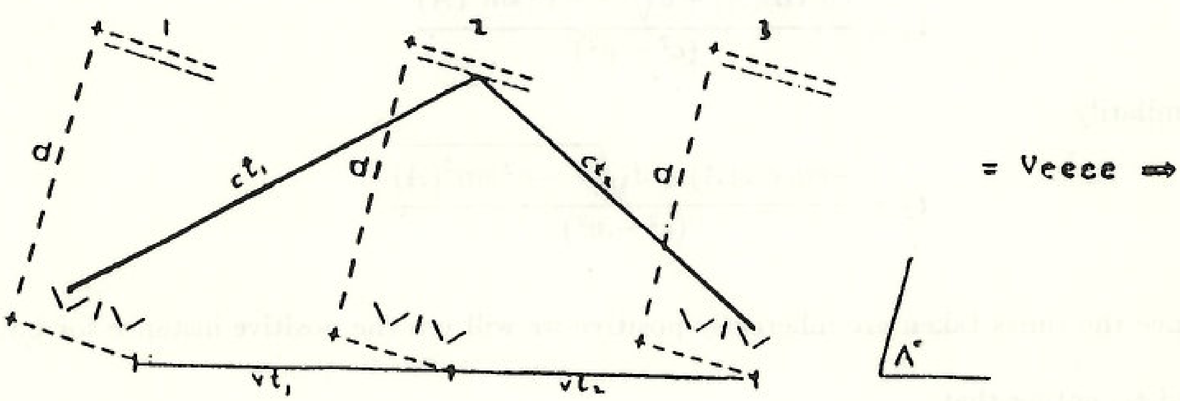
Suppose this device moves past you at velocity "v" and orientated at "A" degrees to the direction of motion.

The following event is timed:

A ray leaves the source, is reflected by the mirror and is detected at the receiver.

The diagram shows the object at three stages in the event

1. When the light is emitted from the source.
2. When it is reflected at the mirror.



The heavy line indicates the path taken by the successful light rays collected at the receiver.

Let t_1 = the time to reach the mirror from the source

t_2 = the time to reach the receiver from the mirror

$t = t_1 + t_2$ the time for the whole transaction.

c = the speed of the photons (light particles)

The distance travelled to the mirror is ct_1 and from the mirror to the receiver is ct_2. The respective horizontal motions of the device are vt_1 and vt_2. Using the cosine rule:

$$c^2t_1^2 = v^2t_1^2 + d^2 + 2vt_1.d.\cos(A)$$

$$(\text{NOTE}: \cos(180 - A) = -\cos(A))$$

$$c^2t_2^2 = v^2t_2^2 + d^2 - 2vt_2.d.\cos(A)$$

and rearranging the equation involving t_1 we have a quadratic equation which is solved to find t_1

$$t_1 = \frac{2vd.\cos(A) + \sqrt{4d^2v^2\cos^2(A) + 4d^2(c^2 - v^2)}}{2(c^2 - v^2)}$$

$$t_1 = \frac{vd.\cos(A) + d\sqrt{c^2 - v^2\sin^2(A)}}{(c^2 - v^2)}$$

Similarily

$$t_2 = \frac{-vd.\cos(A) + d\sqrt{c^2 - v^2\sin^2(A)}}{(c^2 - v^2)}$$

Since the times taken are inherently positive we will use the positive instance for both t_1 and t_2, noting that

$$t = t_1 + t_2$$

the total time for the event is

$$t = \frac{2d\sqrt{c^2 - v^2 \sin^2(A)}}{(c^2 - v^2)}$$

This result indicates that the time taken for the event depends on both its speed and orientation.

Michelson and Morley proposed to detect the motion of the Earth through space by using a rotating configuration of two identical, perpendicular light paths with mirrors at the ends to reflect light from a common source to a receiver at which the interference of the rays could be observed. The time difference for the paths could be derived from the observations of changing interference patterns. No change in the interference patterns was detected. Hence if "t" is measured for my apparatus at various orientations there should be no variation either. Let us assume that this is the case.

Poincare suggested that not only the Michelson Morley experiment, but any experiment would be unable to determine the absolute velocity or direction of motion of an object, this is principle 2.

The second principle indicates that the time "t", recorded on the watch attached to the apparatus, for the event should be the same independent of the velocity of the apparatus, relative to a "stationary" observer, when the event was recorded. Note this does not mean that the value of "t", measured by a stationary observer with a stationary watch, is the same for any value of "v". In fact, the theory of relativity holds that this is not the case.

If we suppose that the value of "t", recorded on the watch attached to the apparatus, does not alter with the orientation or velocity of the device then there is an apparent contràdiction with the algebra. Suppose that we let "d" change with the orientation and velocity of the apparatus so as to allow "t" to be constant. This may seem like a fairly desperate attempt to "fix" the algebra but there are common sense precedents for the idea. For example think of a string of marbles separated by springs, turn it into the flow of a current and the string will shorten. Matter is made up of atoms held by forces and perhaps it may be affected by some cosmic flow coming from a particular direction. Since all measuring sticks would be similarily affected you could not directly measure this alteration of length.

If "d" is to vary then so should "t" so that the speed of light remains constant as supposed in principle 1. What then of our measurements of "t" being constant? Recall that "t" is measured by a watch moving with the apparatus, if the passage of time is altered for the event then so will it be for the watch in such a way that no discrepancy will be detected. The value of "t" may however be different for an observer moving with respect to the apparatus. I will only suppose that the value of "t" depends on the relative velocity of the apparatus but not on its orientation.

To indicate that t depends on v we use $t(v)$ and to indicate that d depends on v and A we use $d(v, A)$

So an hypothesis to reconcile the algebra with the experimental results is to rewrite the equation for t as

$$t(v) = \frac{2d(v,A)\sqrt{c^2 - v^2\sin^2(A)}}{(c^2 - v^2)} \tag{1}$$

or changing the subject

$$d(v,A) = \frac{t(v)(c^2 - v^2)}{2\sqrt{c^2 - v^2\sin^2(A)}} \tag{2}$$

NOTE: Henceforth assume that the general concepts of time t and distance d depending.

on v or A are true for any events and not just the elements of my apparatus (as Poincare did).

Consider events with the same velocity v and orientations of $A = 90$degrees and $A = 0$ degrees in (1).

$$t(v) = \frac{2d(v.90)\sqrt{c^2 - v^2}}{(c^2 - v^2)} \quad \text{and} \quad t(v) = \frac{2d(v.0).c}{(c^2 - v^2)}$$

equating the two expressions and rearranging

$$d(v, 90).\frac{\sqrt{c^2 - v^2}}{c} = d(v, 0),$$

Let

$$L(v) = \frac{\sqrt{c^2 - v^2}}{c}$$

This factor will appear often, it is called the "Lorentz contraction factor". Note that $L(v) < 1$ when $v < c$ and

$$d(v, 90).L(v) = d(v, 0) \tag{3}$$

which means that an observer should see the apparatus shortened in the direction of motion compared to its vertical dimension. Since $L(0) = 1$,

$$d(0, 90) = d(0, 0) \tag{4}$$

which means that orientation has no effect on the distance measurement when the apparatus is "stationary" with respect to the observer.

We suppose that $d(v, 90)$ is constant for all values of v since the object being measured has a velocity of 0 at 90 degrees to the direction of motion, and is therefore "stationary" in this aspect. This means

$$d(v, 90) = d(0, 90) \tag{5}$$

$$\text{and also} \qquad d(v,90) = d(0,0) \qquad \text{using (4).}$$

Considered with (3)

$$d(0,0).L(v) = d(v,0) \tag{6}$$

For $v < c$ the distance measure of an object in the direction of motion is less as measured by a "stationary" observer than the measure made by an observer travelling with the object, or as measured afterwards when the object is "stationary" relative to an observer.

Using (2)

$$d(v,90) = \frac{t(v)(c^2 - v^2)}{2\sqrt{c^2 - v^2}} \text{ and } d(0,90) = \frac{t(0)c^2}{2c}$$

$$d(v,90) = \frac{t(v)\sqrt{c^2 - v^2}}{2} \text{ and } d(0,90) = \frac{t(0)c}{2}$$

Since the left hand sides are equal (using (5)

$$t(0) = L(v)t(v) \tag{7}$$

For $v < c$ the time taken for an event occuring in a moving apparatus is greater as measured by a "stationary" observer than the measure made by an observer travelling with the apparatus, or as measured afterwards when the apparatus is "stationary" relative to an observer. This "time dilation effect" is supported by observations that some short lived particles which result from cosmic collisions in the upper atmosphere live much longer than can be accounted for without supposing that their time rate is slower than ours. It is rather intriguing to realise that our time rate also looks slower to them.

Einstein proposed "The twins paradox" to popularise this effect, but lets use triplets. Two of the triple go on identical journeys in fast spacecraft in opposite directions, eventually returning to the homebound member, each observes the other pair's time to be going slower than their own by virtue of their high relative velocities. Upon returning home the two travellers are equally aged but the homebound one is older. The pair who journeyed, experienced identical accelerations and would have noticed that the account of ages was finalised while decelerating to home. Why is acceleration so kind to those who partake of it?

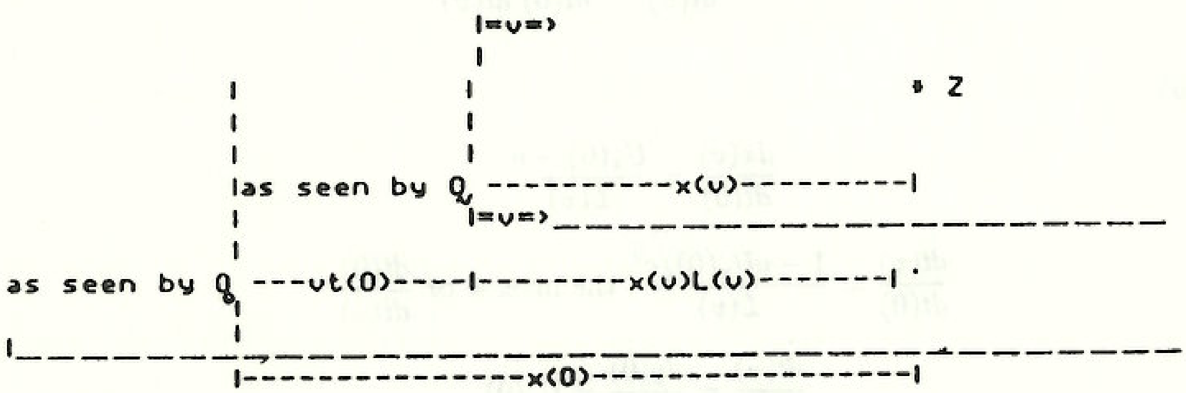
PART 2 Relative Velocities

Suppose there are two observers, one "stationary", named 0_o the other moving at v in the "x" direction relative to 0_o, named 0_v. By previous arrangement, when 0_o and 0_v are at the same position, a light pulse leaves a point Z. 0_o and 0_v each receive the pulse after times of $t(0)$ and $t(v)$ respectively, both at an approach speed of c in accordance to principle 1. 0_o and 0_v calculate the co-ordinates of Z relative to themselves at these times at $(x(0), y(0))$ and $(x(v), y(v))$ respectively.

Hence $\quad x(0) = ct(0) \quad$ and $\quad x(v) = ct(v)$

or $\quad t(0) = x(0)/c \quad$ and $\quad t(v) = x(v)/c \qquad (8)$

In this time 0_o sees 0_v move a distance $vt(0)$ and 0_v is now $x(v)$ units from Z which is seen by 0_o as $x(v).L(v)$ by (6)



Hence

$$x(0) = vt(0) + x(v).L(v) \text{ or } x(v) = \frac{x(0) - vt(0)}{L(v)} \tag{9}$$

The observers now have formulae for calculating the position of Z from the each other's perspective.

Also since there is no motion in the y direction $y(0) = y(v)$

Substituting (8) into (9)

$$ct(v) = \frac{ct(0) - vx(0)/c}{L(v)} \text{ which becomes } t(v) = \frac{t(0) - vx(0)/c^2}{L(v)}$$

Next consider that the point Z is moving with velocity $U_x(0)$ and $U_x(v)$ in the x direction and $U_y(0)$ and $U_y(v)$ in the y direction, relative to 0_o and 0_v respectively

that $\frac{dx(0)}{dt(0)} = U_x(0)$ and $\frac{dx(v)}{dt(v)} = U_x(v)$

How are the velocities of Z related? In Newtonian Physics $U_x(v) = U_x(0) - v$ but now things are different.

$$U_x(v) = \frac{dx(v)}{dt(v)} = \frac{dx(v)}{dt(0)}\frac{dt(0)}{dt(v)}$$

$$U_y(v) = \frac{dy(v)}{dt(v)} = \frac{dy(v)}{dt(0)}\frac{dt(0)}{dt(v)} \tag{10}$$

from (9)

$$\frac{dx(v)}{dt(0)} = \frac{U_x(0) - v}{L(v)}$$

$$\frac{dt(v)}{dt(0)} = \frac{1 - vU_x(0)/c^2}{L(v)} \text{ the inverse of } \frac{dt(0)}{dt(v)}$$

$$\frac{dt(v)}{dt(0)} = \frac{dy(0)}{dt(0)} = U_y(0)$$

NOTE: In the y direction $U_x(0) = dx(0)/dt(0) = 0$. Also, since v is constant, $L(v)$ is a constant with respect to time.

Hence (10) becomes

$$U_x(v) = \frac{U_x(0) - v}{1 - U_x(0)v/c^2} \quad \text{and} \quad U_y(v) = U_y(0).L(v) \tag{11}$$

This result indicates that objects externally observed as approaching at a combined speed in excess of c will appear to each other to be approaching at less that c. In particular if the approach speed $U_x(0) = -c$ then $U_x(v)$ also is c, i.e. principle 1.

PART 3 Mass and energy

There are three assumptions that are retained from Newtonian Physics: mass, energy and momentum are conserved in a moving system just as they are in a "stationary" one. Let's consider the momentum of a ball as it moves in the "Y" direction. The momentum can be assessed by measuring its mass beforehand then timing it as it moves between two points to determine velocity which we will suppose is negligible compared to "c". For an observer moving past in the "X" direction (at 90 degrees to "Y") the "Y" component of the momentum of the ball should appear the same as for the observer sitting still relative to the measuring points. Assuming that both observers determine identical "Y" momentum for the ball and considering that the moving observer sees the ball move slower due to time dilation and yet is expected to calculate that its momentum is unaffected! (see (11)). This is apparently another inconsistency. Suppose an object has mass m and y velocity of $U_y(v)$ (which is also equal to $U_y(0).L(v)$ by (11), its momentum in the Y direction is the same for both observers.

i.e. $\quad m.U_y(v) = m.U_y(0) \quad$ but $\quad U_y(v) = U_y(0).L(v)$

Hence to "fix" the contradiction, we follow the precedent for time and suppose m depends on v, i.e., let

$$m(v) = m(0)/L(v) \tag{12}$$

For $v < c$ the mass of an object is greater as measured by a "stationary" observer than the measure made by an observer travelling with the object, or as measured afterwards when the object is "stationary" relative to an observer. Note that as speed builds towards c, the object's mass increases without bound, hence no object with non zero mass can be given sufficient energy to reach the speed of light.

Using (12) $m(v)^2 = \dfrac{m(0)^2}{L(v)} = \dfrac{m(0)^2}{c^2}\dfrac{c^2}{v^2}$ substituting for $L(v)$

which is modified to

$$m(v)^2c^2 = m(0)^2c^2 + m(v)^2v^2$$

where $m(v).v$ is the momentum, commonly called "P". Hence

$$m(v)^2c^2 = m(0)^2v^2 \tag{13}$$

Expanding $1/L(v)$ by Taylor series, (12) becomes

$$m(v) = m(0)[1 + v^2/2c^2 + 3m(0)v^2/8c^4 = \cdots \tag{14}$$

The units of both sides of (14) are those of energy and the familiar Newtonian Kinetic energy term appears second on the right hand side of the equation. The terms involving "v" can be equated by $P^2/m(v)$ using (13). Einstein proposed that (14) was in fact an energy equation and that the total energy of a particle is $m(v)c^2$.